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REFLECTION AND REFRACTION OF FINITE AMPLITUDE ACOUSTIC WAVES AT A FLUID-FLUID INTERFACE

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Presented in this dissertation is a theoretical investigation of the nonlinear effects in reflection and refraction of plane finite-amplitude acoustic waves at an initially plane interface between two lossless fluids. The dissertation is divided into three parts. In the first part, the terms in the equations of motion for a homogeneous, thermoviscous fluid with a single relaxation mechanism are rank-ordered to determine the most important nonlinear and dissipation terms. The equations are then combined to form a general wave equation that includes the most important effects of nonlinearity and dissipation. In the second part, the terms in the boundary conditions between two *lossless* fluids are rank-ordered to include the most important nonlinear effects. Subject to these boundary conditions, a solution of the *lossless* form of the aforementioned wave equation is obtained by way of second-order perturbation expansion. The lossless form of the wave equation and the lossless boundary conditions are expanded, and the $O(\epsilon)$ and $O(\epsilon^2)$ systems are solved in terms of a modified velocity potential. The analysis is performed for oblique incidence, and the

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19. boundary condition at the source is arbitrary. Effects examined include the finite displacement of the interface and the variation of the direction of the normal to the interface, both of which are caused by the motion of the interface as it responds to the incident sound. In the third part, two different modified forms of Snell's law for the special case of simple wave flow are derived: one by matching the trace velocities at the interface and one by matching the variation of the pressure along the interface. It is shown, however, that to $O(\epsilon^2)$, both reduce to ordinary Snell's law and a condition necessary for simple wave flow.

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PREFACE

This report is an adaptation of the Ph.D. dissertation of the same title by Frederick D. Cotaras. Mr. Cotaras was enrolled in the Department of Electrical and Computer Engineering, and will receive his degree in May 1989.

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CHAPTER 1

INTRODUCTION

1–1 Motivation and Previous Research

A theoretical investigation of the nonlinear effects in the reflection and refraction of plane finite-amplitude acoustic waves at the interface between two lossless fluids is presented in this dissertation. This research may in one sense be considered an outgrowth of the author's master's thesis work (Cotaras 1985), which dealt with finite-amplitude effects in long range underwater acoustic propagation. That work is rigorously valid only up to the point that the acoustic waves interact with either the surface or the bottom of the ocean or pass through a *caustic* (focal region). Although this work does not contain any discussion of what happens at a caustic, it does have application to the reflection and refraction of finite-amplitude sound at both the surface and the bottom of the ocean.

Evidence from both inside and outside the field of nonlinear acoustics indicates that the laws governing the reflection and refraction of finite-amplitude sound may be different from the well-known linear theory results—Snell's law and the law of specular reflection. From outside the field of acoustics come the unusual effects that are observed during the reflection and refraction of plane shock waves, effects such as the stem shock.[1] These effects have been studied for some time and are still of interest today [see, for example, Polachek and Seeger (1951); Jahn (1956, 1957); Henderson (1966); Miura and Glass (1983)]. The results of the research are not, however, of direct use in acoustics for the following reasons: (1) The pressure change at the shock is typically much larger than that in finite-amplitude acoustics, and (2) the results indicate only what happens at the shock, not what happens to a continuous signal. From within the field of nonlinear

[1] Upon reflection from an interface, the intersection point of the incident and reflected shocks detaches from the interface. The intersection point is, however, still 'connected' to the interface by a third shock—the stem shock. For more details, see, for example, Courant and Friedrichs (1976, p. 334).

acoustics, evidence of a possible difference from linear theory is suggested by Blackstock (1959). Studying the normal reflection of finite-amplitude sound from a rigid wall, Blackstock noted that the linear theory result of pressure doubling is not rigorously correct. Pressure doubling is actually the small-signal limit of the proper law, which is that excess sound speed $(c - c_0)$ doubles.[2]

Despite the evidence of a possible deviation from the linear theory, the problem of reflection and refraction of finite-amplitude sound has remained almost untouched. Van Buren and Breazeale (1968) dealt with the reflection of finite-amplitude ultrasonic waves by decomposing the incident finite-amplitude signal into its Fourier components and then reflecting each component independently using linear theory. In their procedure the reflected components are then recombined to reconstitute the reflected wave. Thus they did not directly address the question of nonlinear effects in reflection and refraction.

Two Chinese reseachers have recently published pertinent work: Qian (1982) on reflection of finite-amplitude acoustic waves and Feng (1983) on reflection and refraction. Both use perturbation methods and Lagrangian coordinates. Qian examined that which is a special case for us—the reflection of finite-amplitude sound from a rigid wall at oblique incidence. The paper has a few algebraic errors[3] but is otherwise correct for the case it treats. On the other hand, the work by Feng appears to contain some fundamental errors in both the wave equations and the boundary conditions. After expanding all the acoustic variables in a small parameter ϵ, Feng obtains the $O(\epsilon)$ and $O(\epsilon^2)$ wave equations and boundary conditions. However, Feng then assumes that the interaction terms in the $O(\epsilon^2)$ wave equations [his Eqs. (13) and (15)] are zero. This is true only for the case of simple wave motion.[4] For compound wave flow, which is what occurs on the fluid I side (the incident and reflected fields co-exist), the terms do not cancel. Feng's wave equations are, therefore, not valid on the fluid I side. Furthermore, in his $O(\epsilon^2)$ particle velocity boundary condition, Feng fails to account for the variation of the normal to the interface. It is shown in this dissertation that the variation vanishes only in the special case of normal incidence.

An earlier presentation of some of the results of this research (Cotaras and Blackstock 1987) apparently stimulated some interest in the problem. For

[2]For finite-amplitude waves, the sound speed c differs from the small-signal sound speed c_0: see the next section.

[3]For example, in Eq. (13a) an unneccesary $1/(c_0^{\mathrm{I}})^2$, where c_0^{I} is the small-signal sound speed in fluid I, is introduced; this error is repeated in Eq. (19).

[4]For the case of plane waves, simple wave flow may be defined as a wave field that consists of progressive waves only, that is, waves propagating in one direction only. If progressive and regressive waves co-exist, then the wave field is referred to as compound.

example, Shu and Ginsberg (1988) are examining the reflection and refraction of finite-amplitude sound at a fluid-solid interface. At the time of writing of this dissertation, however, their work was not available for examination.

1–2 The Propagation Speed of Finite-Amplitude Acoustic Waves[5]

It is well known that the propagation of a finite-amplitude wave cannot be accurately modeled by a small-signal acoustical theory; see, for example, Blackstock (1972). Small-signal theory fails because the propagation speed of a finite-amplitude wave depends on the local particle velocity. The dependence arises in two ways. (1) Convection: Convection occurs when the fluid particles themselves are set into motion by the passing acoustic wave and contribute their own velocity to the total wave speed. Thus the actual propagation speed dx/dt may be expressed as

$$\frac{dx}{dt} = c + u \quad , \tag{1.1}$$

where c is the sound speed (which is to be distinguished from the propagation speed). (2) The nonlinearity of the pressure-density relationship: Because of this nonlinearity, the sound speed c is not, in general, a constant. For an outgoing plane wave, the sound speed is given by the following [see, for example, Beyer (1974)]:

$$c = c_0 + \frac{B}{2A} u \quad , \tag{1.2}$$

where c_0 is a true constant referred to as the small-signal sound speed (the value that appears in tables), and A and B are the first and second coefficients of the Taylor series expansion of the pressure density relation. When the above equations are combined, the resulting equation is usually written as

$$\frac{dx}{dt} = c_0 + \beta u \quad , \tag{1.3}$$

where $\beta \equiv 1 + B/2A$ is called the coefficient of nonlinearity and has the value 1.2 for air and 3.5 for water. Equation (1.3) is sometimes called the *wavelet* speed, a wavelet being a given point on the waveform of a propagating wave.

[5]For a more thorough introduction to the physical origins of nonlinear acoustics and its applications, see, for example, Hamilton (1986).

1–3 Scope of the Research

The primary objective of this dissertation is to investigate nonlinear effects in reflection and refraction at a plane interface between two lossless fluids. However, because losses are present in all real fluids and because losses must eventually be accounted for in a complete treatment of reflection and refraction, the equation for finite-amplitude wave motion in a homogeneous, thermoviscous fluid with a single relaxation mechanism is derived in this dissertation. Although thermoviscous effects, relaxation effects, and nonlinear effects have been studied separately, the derivation (from first principles) of the wave equation that accounts for all three effects has not been previously attempted. It is therefore included in this dissertation.

The dissertation is divided into three parts. Presented in the first part are the basic equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism and a method for ranking the terms in the equations. The fluid is assumed to be initially quiet, irrotational, uniform, and in thermodynamic equilibrium. Moreover, effects of rotational flow are ignored. When presenting the basic equations, we make a fundamental assumption: The magnitude of the deviation from thermodynamic equilibrium is assumed to be small, and linear relationships between each thermodynamic flux and all the thermodynamic forces are assumed to hold. The possibility of cross-effects between the different thermodynamic fluxes and forces is pointed out, but neglected. Ranking the terms in the basic equations according to their relative importance requires that additional fundamental assumptions be made about (1) the amplitude of the acoustic signal, (2) the magnitude of the transport coefficients, and (3) the magnitude of the dispersion caused by the relaxation. Use of the ranking system enables us to develop simplified forms of the equations. The wave equation for finite-amplitude signals in a thermoviscous fluid with a single relaxation mechanism is then developed.

Analyzed in the second part of this dissertation is the reflection and refraction of finite-amplitude plane waves that are obliquely incident on an initially plane interface between two lossless, immiscible fluids. First, the boundary conditions at the interface are examined. The interface is assumed to be initially planar and coincident with the $z = 0$ plane. Moreover, the effects of surface tension and body forces at the interface are neglected. Second-order perturbation analysis, which is sometimes referred to as quasilinear analysis, is employed to analyze reflection and refraction of an abitrary incident signal. The notation closely follows the work of Naze Tjøtta and Tjøtta (1987). The $O(\epsilon)$ and $O(\epsilon^2)$ systems, where ϵ is a small parameter, are then solved. The

$O(\epsilon^2)$ system accounts for not only the nonlinearity in the wave equation but for the finite displacement of the interface and the variation of the normal to the interface. After the expressions for the $O(\epsilon)$ and $O(\epsilon^2)$ reflected and transmitted fields are obtained, it is noted that, to $O(\epsilon^2)$, Snell's law and the law of specular reflection hold.

In the third part of the dissertation, we develop two different 'modified forms' of Snell's law—forms which appear to indicate that refraction has a slight amplitude dependence. The 'modified forms' are developed by means other than perturbations and appear to be correct to second-order. However, in deriving the 'modified forms', one of which was reported by Cotaras and Blackstock (1987), a large assumption is made: that simple wave flow exists on the incident side of the interface. This assumption is quantified by using the results of the previous section to develop the $O(\epsilon)$ and $O(\epsilon^2)$ conditions for simple wave flow. We next expand each of the 'modified forms' of Snell's law and find that, to $O(\epsilon^2)$, the two 'modified forms' are equivalent. Moreover, the 'modified forms' yield, at $O(\epsilon)$, ordinary Snell's law and, at $O(\epsilon^2)$, one of the $O(\epsilon^2)$ conditions for simple wave flow. Since simple flow is assumed in the derivation, the 'modified forms' are, to $O(\epsilon^2)$, equivalent to ordinary Snell's law. Thus we metaphorically deflate the balloon that carried us through this dissertation: the hope of finding a deviation from Snell's law at $O(\epsilon^2)$.

CHAPTER 2

ON THE RANKING OF TERMS IN THE BASIC EQUATIONS FOR A HOMOGENOUS, THERMOVISCOUS FLUID WITH A SINGLE RELAXATION MECHANISM

Discussed in this chapter are the basic equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism and a method for ranking the terms in the equations. The equations presented are the conservation of mass (continuity equation), the conservation of linear momentum (Newton's second law), and the conservation of entropy. To the entropy equation, however, we add the second law of thermodynamics: entropy production is positive for irreversible processes (loss mechanisms). Other relations from thermodynamics that are presented are an equation of state, Gibb's equation, and the first law of thermodynamics: the conservation of energy. The energy equation is manipulated into the form of the entropy equation, and entropy production is seen to be the sum of the products of the various thermodynamic *fluxes* and *forces*. A fundamental assumption is then made. The magnitude of the deviation from thermodynamic equilibrium is assumed to be small, and linear relationships between each thermodynamic flux and all the thermodynamic forces are assumed to hold. The possibility of cross-effects between the different thermodynamic fluxes and forces is pointed out, but neglected. The fluid is assumed to be initially quiet and in thermodynamic equilibrium. Moreover, the signal is assumed to be far from all boundaries for all time. Additional fundamental assumptions are then made about (1) the amplitude of the acoustic signal, (2) the magnitude of the transport coefficients, and (3) the magnitude of the dispersion caused by the relaxation. The terms in the equations are then ranked according to their relative importance. Use of the ranking system enables us to readily obtain simplified forms of the basic equations (Lighthill 1956). Specifically, we develop simplified forms of the equations appropriate for (1) small-signal wave motion in a lossless fluid, (2) small-signal wave motion in a homogeneous, thermoviscous, relaxing fluid, and (3) finite-amplitude wave motion in a homogeneous, thermoviscous,

relaxing fluid. As mentioned in Chapter 1, it is our intention to neglect all losses in the analysis presented in later chapters. The terms that represent the effects of viscosity, heat conduction, and relaxation are, however, included at this stage for completeness.

2–1 Basic Equations for a Homogeneous, Thermoviscous Fluid with a Single Relaxation Mechanism

We start with two classical equations: the conservation of mass and the conservation of linear momentum,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0 \quad , \tag{2.1}$$

$$\rho \frac{Du_k}{Dt} = \frac{\partial \sigma_{ik}}{\partial x_i} \quad , \tag{2.2}$$

where ρ is the density, u_k is the kth component of the particle velocity $\mathbf{u}$ ($k = 1, 2, 3$), σ_{ik} is the stress tensor, and D/Dt is the material derivative,

$$\frac{D(\cdot)}{Dt} \equiv \frac{\partial(\cdot)}{\partial t} + u_i \frac{\partial(\cdot)}{\partial x_i} \quad . \tag{2.3}$$

Throughout this work, we follow the summation convention; that is, a repeated index implies a summation over the index. Equations (2.1) and (2.2) are from Thompson [1984, Eqs. (1.54) and (1.59)], but the effect of body forces such as gravity is neglected in the linear momentum equation.

To the pair of classical equations, we add some thermodynamic relations: a state equation, Gibb's equation, and the entropy equation. The state equation gives the functional relationship between the thermodynamic variables *when the fluid is in thermodynamic equilibrium.* (Thermodynamic equilibrium is precisely defined below.) For a thermoviscous fluid with a single relaxation mechanism, one form of the state equation is[1]

$$P = P(\rho, s, q) \quad , \tag{2.4}$$

[1]Any three thermodynamic variables may be chosen as the three independent variables. The advantage of choosing the advancement variable q and the entropy s as two of the three is that, if a fluid is always in thermodynamic equilibrium (very common assumption in acoustics), both q and s take on their equilibrium values, which are constants. Thus, the state equation reduces to $P = P(\rho)$.

where P is the total pressure, s is the entropy per unit mass, and q is the degree of advancement of the relaxation mechanism, for example, a chemical reaction. It turns out that the following alternate state equation is required:

$$T = T(\rho, s, q) \quad , \tag{2.5}$$

where T is the absolute temperature. Another relation between the thermodynamic variables is given by Gibb's equation, which for a fluid with a single relaxation mechanism is

$$T ds = de - \frac{P}{\rho^2} d\rho - A\, dq \quad , \tag{2.6}$$

where e is the internal energy per unit mass and A is the affinity of the relaxation process. The entropy equation may be stated in such a way that it incorporates the second law of thermodynamics:

$$\rho \frac{Ds}{Dt} + \frac{\partial J_k}{\partial x_k} = \Upsilon \quad , \qquad \Upsilon \geq 0 \tag{2.7}$$

where J_k is the kth component of the entropy flux vector $\mathbf{J}$ and Υ is the entropy source strength.[2]

The entropy source strength is an important variable for two reasons. First, the fluid is defined as being in a state of thermodynamic equilibrium when no entropy is produced, that is, when $\Upsilon = 0$. Second, linear phenomenological relations may be introduced by way of the entropy source strength. The entropy source strength may be written as the sum of the product of thermodynamic fluxes and forces,

$$\Upsilon = J_k X_k \quad , \tag{2.8}$$

where J_k is a generalized thermodynamic flux and X_k is a generalized thermodynamic force. Since thermodynamic equilibrium is defined as zero entropy production, it is required that all thermodynamic fluxes and forces be zero at equilibrium.

$$J_k = 0 \text{ and } X_k = 0 \text{ at equilibrium} \quad . \tag{2.9}$$

If the departures from equilibrium are *small*, linear phenomenological relations between the thermodynamic fluxes and forces may be assumed,

$$J_k = L_{ik} X_i \quad , \tag{2.10}$$

[2]The material in this section is based on the material in Chaps. III, IV, and V in the book by Prigogine (1961) and Chaps. III, IV, and VI in the book by de Groot and Mazur (1984). References to other works are made when appropriate.

where L_{ik} are called the phenomenological coefficients. Note that although a linear relationship between the fluxes and forces is assumed, the coefficients are not assumed to be constants, and the possibility of interference between the different fluxes and forces is permitted. The interference between the thermodynamic fluxes and forces is, however, restricted by Curie's principle, which, briefly stated, is that for an isotropic fluid only fluxes and forces of similar tensorial nature may interact. Thus, a scalar flux may contain contributions from scalar forces, but not from vector forces or tensor forces. Moreover, the number of coefficients is limited by Onsager's relation, which states that $L_{ik} = L_{ki}$.

Finding the appropriate form of Υ

To find the form of Υ appropriate for a thermoviscous fluid with a single relaxation mechanism, we start with the energy equation and rearrange it so that it is in the form of the entropy equation. In rearranging the equation, we assume *local* equilibrium and split the stress tensor into two parts, a pressure term and a viscous stress term.

We start with the energy equation; see, for example, Thompson [1984, Eq. (1.60)],

$$\rho \frac{D}{Dt}\left(e + \frac{u^2}{2}\right) = \frac{\partial}{\partial x_i}(\sigma_{ik} u_k) - \frac{\partial Q_i}{\partial x_i} \quad , \tag{2.11}$$

where Q_i is the ith component of the heat flux vector $\mathbf{Q}$. If the kinetic energy relation (which is obtained by multiplying the linear momentum equation by u_k) is subtracted from the energy equation, the resulting equation is [see, for example, Thompson (1984, Eq. 1.70)]

$$\rho \frac{De}{Dt} = \sigma_{ik} \frac{\partial u_i}{\partial x_k} - \frac{\partial Q_i}{\partial x_i} \quad . \tag{2.12}$$

The stress tensor σ_{ik} may be divided into two parts:

$$\sigma_{ik} = -P\delta_{ik} - \sigma'_{ik} \quad , \tag{2.13}$$

where σ'_{ik} is the viscous stress tensor, $-P\delta_{ik}$ is the pressure stress, and δ_{ik} is the Kronecker delta function, which is equal to unity if $i = k$, but zero otherwise.[3]

[3]The choice of sign on the viscous stress tensor in Eq. (2.13), which is arbitrary, differs from some work in hydrodynamics [see, for example, Landau and Lifshitz 1959, Eq. (15.2)], but conforms to some work in thermodynamics [see, for example, de Groot and Mazur 1984, Eq. (III.35)].

The viscous stress tensor is assumed to be symmetric.[4] Using Eq. (2.13) in Eq. (2.12) and using the continuity equation, Eq. (2.1), leads to the following form of the energy equation:

$$\frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} = -\frac{1}{\rho}\sigma'_{ik}\frac{\partial u_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial Q_i}{\partial x_k} \quad . \tag{2.14}$$

Equation (2.14) may be simplified using Gibb's equation. However, Gibb's relation is, strictly speaking, only valid in equilibrium. Since the fluid is not always in equilibrium, we must assume that the fluid particles are in a state of *local* equilibrium. If we follow a fluid particle that is in local equilibrium, Gibb's equation may be written as

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} - A\frac{Dq}{Dt} \quad . \tag{2.15}$$

Use of Eq. (2.15) in Eq. (2.14) yields

$$\rho\frac{Ds}{Dt} = -\frac{1}{T}\sigma'_{ik}\frac{\partial u_i}{\partial x_k} - \frac{1}{T}\frac{\partial Q_i}{\partial x_k} - \frac{\rho A}{T}\frac{Dq}{Dt} \quad . \tag{2.16}$$

To place Eq. (2.16) in the form of the entropy equation, Eq. (2.7), we must perform some manipulations that initially appear arbitrary. The resulting expressions are, however, unique; see de Groot and Mazur (1984, pp. 24–25). The simpler of the manipulations is

$$\frac{1}{T}\frac{\partial Q_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{Q_i}{T}\right) + \frac{Q_i}{T^2}\frac{\partial T}{\partial x_i} \quad . \tag{2.17}$$

The more complex manipulation involves expanding σ'_{ik} and $\partial u_i/\partial x_k$ into trace components and remainders that have zero trace. We rewrite the viscous stress tensor as follows:

$$\sigma'_{ik} = \breve{\sigma}'_{ik} + \tfrac{1}{3}\delta_{ik}\sigma'_{jj} \quad , \tag{2.18}$$

where $\breve{\sigma}'_{ik}$ is a symmetric tensor since σ'_{ik} is assumed symmetric. The gradient of the velocity is rewritten as

$$\frac{\partial u_i}{\partial x_k} = \frac{1}{2}\left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right] + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right) + \frac{1}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell} \quad .$$

[4] This is tantamount to assuming that the internal angular momentum is randomly distributed amongst the molecules, which is generally the case. If this is not assumed, then it turns out that another phenomenological coefficient, in this case, the rotational viscosity, is required. Moreover, the asymmetry of the viscous stress tensor decays; see de Groot and Mazur (1984, Chap. XII).

The first term on the right-hand side, which is symmetric and has zero trace, indicates shearing motion but without a volume change. The second term, which is antisymmetric, is associated with a rotation but again without a volume change. The third term on the right-hand side, which consists solely of diagonal elements, indicates the volume change. Since the product of symmetric and antisymmetric tensors is zero, we may write

$$\sigma'_{ik}\frac{\partial u_i}{\partial x_k} = \frac{\breve{\sigma}'_{ik}}{2}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)-\tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right]+\tfrac{1}{3}\sigma'_{jj}\frac{\partial u_\ell}{\partial x_\ell} \quad . \tag{2.19}$$

Use of Eqs. (2.17) and (2.19) in Eq. (2.16) yields the desired form of the energy equation:

$$\rho\frac{Ds}{Dt}-\frac{\partial}{\partial x_i}\left(\frac{Q_i}{T}\right) = -\frac{Q_i}{T^2}\frac{\partial T}{\partial x_i}-\frac{\breve{\sigma}'_{ik}}{2T}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)-\tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right]$$
$$-\frac{1}{3}\frac{\sigma'_{jj}}{T}\frac{\partial u_\ell}{\partial x_\ell}-\frac{\rho A}{T}\frac{Dq}{DT} \quad . \tag{2.20}$$

Comparing Eq. (2.20) with the entropy equation, Eq. (2.7), we identify the following:

$$J_i = \frac{Q_i}{T} \tag{2.21}$$

and

$$\Upsilon = -\frac{Q_i}{T^2}\frac{\partial T}{\partial x_i}-\frac{\breve{\sigma}'_{ik}}{2T}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)-\tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right]-\frac{1}{3}\frac{\sigma'_{jj}}{T}\frac{\partial u_\ell}{\partial x_\ell}-\frac{\rho A}{T}\frac{Dq}{Dt} \quad . \tag{2.22}$$

Introduction of the linear phenomenological relations and constants

We are now in a position to utilize Eqs. (2.8) and (2.10), the linear phenomenological equations. Recalling Curie's principle and Onsager's relations, we write

$$Q_i = -\frac{L_1}{T^2}\frac{\partial T}{\partial x_i} \quad , \tag{2.23}$$

$$\breve{\sigma}'_{ik} = -\frac{L_2}{2T}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)-\tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right] \quad , \tag{2.24}$$

$$\tfrac{1}{3}\sigma'_{jj} = -\frac{L_{11}}{T}\frac{\partial u_\ell}{\partial x_\ell}-L_{12}\frac{A}{T} \quad , \tag{2.25}$$

$$\rho\frac{Dq}{Dt} = -\frac{L_{21}}{T}\frac{\partial u_\ell}{\partial x_\ell}-L_{22}\frac{A}{T} \quad , \tag{2.26}$$

where

$$L_{12} = L_{21} \quad . \tag{2.27}$$

The coefficients L_1, L_2, L_{11}, and L_{22} must be positive in order to satisfy the second law of thermodynamics, $\Upsilon \geq 0$. For simplicity, we shall neglect the possiblity of cross-effects throughout the remainder of the work, that is, we assume that $L_{12} = L_{21} = 0$.

We now identify some of the phenomenological coefficients as the transport coefficients in well-known linear relations (for example, Fourier's law of heat conduction):

$$\kappa = \frac{L_1}{T^2} \quad , \tag{2.28}$$

$$\mu = \frac{L_2}{2T} \quad , \tag{2.29}$$

$$\mu_{\mathrm{B}} = \frac{L_{11}}{T} \quad , \tag{2.30}$$

where κ is the thermal conductivity, μ is the shear viscosity, and μ_{B} is the bulk viscosity. Use of Eqs. (2.28)–(2.30) in Eqs. (2.23)–(2.25) yields the following well-known linear phenomenological laws (recall that the cross-effect in Eq. (2.25) is neglected):

$$\mathbf{Q} = -\kappa \boldsymbol{\nabla} T \quad , \tag{2.31}$$

$$\breve{\sigma}'_{ik} = \mu \left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) - \tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell} \right] \quad , \tag{2.32}$$

$$\tfrac{1}{3}\sigma'_{jj} = \mu_{\mathrm{B}}(\boldsymbol{\nabla}\cdot\mathbf{u}) \quad . \tag{2.33}$$

Obtaining the fourth phenomenological constant and relation takes a little more work. First, for future reference, we redefine the phenomenological coefficient as

$$\beta' = \frac{L_{22}}{\rho T} \quad . \tag{2.34}$$

If cross-effects are neglected, this definition results in the following linear phenomenological relation:

$$\frac{Dq}{Dt} = -\beta' A \quad . \tag{2.35}$$

It turns out, however, that different forms of the linear phenomenological relation and its associated coefficient, which are referred to as the rate equation and the relaxation time, are of great use. The rate equation and the relaxation time are obtained as follows: The affinity $A = A(\rho, T, q)$ is expanded in a Taylor series.

The expansion is then specialized for equilibrium, and the result is subtracted from the original expansion. The results follow directly. In general, for *small* deviations from equilibrium, A may be approximated as

$$A = \left(\frac{\partial A}{\partial \rho}\bigg|_{T,q} \right)_0 (\rho - \rho_0) + \left(\frac{\partial A}{\partial T}\bigg|_{\rho,q} \right)_0 (T - T_0) + \left(\frac{\partial A}{\partial q}\bigg|_{\rho,T} \right)_0 (q - q_0) \quad , \tag{2.36}$$

where the subscript 0 denotes the static value of the variable. (This notation is used throughout this work.) As noted earlier, all thermodynamic forces and fluxes are zero at equilibrium; thus, the equilibrium—static—value of A is zero. The equilibrium value of the advancement variable q is denoted by a superscript $*$.

$$A\big|_{q=q^*} = 0 = \left(\frac{\partial A}{\partial \rho}\bigg|_{T,q} \right)_0 (\rho - \rho_0) + \left(\frac{\partial A}{\partial T}\bigg|_{\rho,q} \right)_0 (T - T_0) + \left(\frac{\partial A}{\partial q}\bigg|_{\rho,T} \right)_0 (q^* - q_0^*) \quad , \tag{2.37}$$

Equation (2.37) indicates that q^* is a function of only two thermodynamic variables, ρ and T. Note that the static—equilibrium—values of q and q^* are equal, that is,

$$q_0 = q_0^* \quad .$$

Subtracting Eq. (2.37) from Eq. (2.36) yields

$$A = \left(\frac{\partial A}{\partial q}\bigg|_{\rho,T} \right)_0 (q - q^*) \quad . \tag{2.38}$$

This is consistent with our earlier assumption that the thermodynamic fluxes and forces be linearly related. Use of Eq. (2.38) in the linear phenomenological relation (again neglecting cross-effects), Eq. (2.26), leads to

$$\frac{Dq}{Dt} = -\frac{(q - q^*)}{\tau_0} \quad , \tag{2.39}$$

where

$$\tau_0 \equiv \frac{\rho T}{L_{22} \left(\frac{\partial A}{\partial q}\Big|_{\rho,T} \right)_0} \quad . \tag{2.40}$$

The value of τ_0 is positive because L_{22} must be positive to satisfy the second law, $\Upsilon \geq 0$, and because $\left(\frac{\partial A}{\partial q}\Big|_{\rho,T} \right)_0$ is positive; see de Groot and Mazur (1984, p. 201) for a complete explanation.

Introduction of the linear phenomenological relations into the momentum and entropy equations

The terms in the entropy equation, Eq. (2.7), that were unknown are the entropy flux, $\mathbf{J}$, and the entropy source strength, Υ. The entropy flux is given in Eq. (2.21). The entropy source strength may be obtained if the linear relations between the thermodynamic forces and fluxes given in Eqs. (2.31), (2.32), (2.33), and (2.35) are substituted into the relation for Υ, Eq. (2.22),

$$\Upsilon = \frac{\kappa}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\mu}{2T}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right]^2 + \frac{\mu_{\mathrm{B}}}{T}\left(\frac{\partial u_\ell}{\partial x_\ell}\right)^2 + \frac{\beta' \rho}{T}A^2 \quad . \tag{2.41}$$

Note that all the terms are positive and the squares of variables.

The linear phenomenological relations are now introduced into the momentum equation. Recall that the viscous stress tensor was split into two parts: a trace component and a symmetric tensor with zero trace. Use of Eqs. (2.32) and (2.33) in Eq. (2.18) yields

$$\sigma'_{ik} = -\mu\left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right] - \mu_{\mathrm{B}}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell} \quad . \tag{2.42}$$

Use of Eqs. (2.13) and (2.42) in Eq. (2.2) yields the following form of the linear momentum equation:

$$\rho\frac{Du_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial u_i}{\partial x_k \partial x_k} + \left(\mu_{\mathrm{B}} + \frac{\mu}{3}\right)\frac{\partial}{\partial x_i}\frac{\partial u_\ell}{\partial x_\ell} \quad , \tag{2.43}$$

which may also be written in vector form as

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}P + \mu\nabla^2\mathbf{u} + \left(\mu_{\mathrm{B}} + \frac{\mu}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) \quad . \tag{2.44}$$

In obtaining Eq. (2.43), we assumed that the transport coefficients κ, μ, and μ_{B} are constants.[5] Equation (2.44) is the well-known form of the momentum equation; see Landau and Lifshitz [1959, Eq. (15.6)].

[5] This assumption is, strictly speaking, not necessary; it is, however, convenient. It is not necessary because it turns out that the ranking scheme, which is introduced later in the chapter, indicates that the terms which arise if the transport coefficients are not assumed constant are negligible. As an example, we now examine one of the terms containing a transport coefficient, $\mu_{\mathrm{B}}\delta_{ik}\partial u_\ell/\partial x_\ell$. The gradient of this term is

$$\boldsymbol{\nabla}(\mu_{\mathrm{B}}\boldsymbol{\nabla}\cdot\mathbf{u}) = \mu_{\mathrm{B}}\nabla^2\mathbf{u} + (\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\nabla}\mu_{\mathrm{B}} \quad .$$

But since μ_{B} is a function of any three state variables, say P, s, and q, we may expand μ_{B} in

Summary of basic equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism

The basic equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism are the continuity equation, the linear momentum equation, the entropy equation, the equation of state, the alternate equation of state, and the rate equation. The equations are, respectively,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{u}) = 0 \quad , \tag{2.1}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\mu_B + \frac{\mu}{3}\right)\nabla(\nabla\cdot\mathbf{u}) \quad , \tag{2.44}$$

$$\begin{aligned}\rho\frac{Ds}{Dt} - \kappa\nabla\cdot\left(\frac{\nabla T}{T}\right) = \frac{\kappa}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 &+ \frac{\mu}{2T}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik}\frac{\partial u_\ell}{\partial x_\ell}\right]^2 \\ &+ \frac{\mu_B}{T}\left(\frac{\partial u_\ell}{\partial x_\ell}\right)^2 + \frac{\beta'\rho}{T}A^2 \quad ,\end{aligned} \tag{2.45}$$

$$P = P(\rho, s, q) \quad , \tag{2.4}$$

$$T = T(\rho, s, q) \quad , \tag{2.5}$$

$$\frac{Dq}{Dt} = -\frac{(q - q^*)}{\tau_0} \quad . \tag{2.39}$$

2–2 The Ranking System and Its Fundamental Assumptions

In this section the thermodynamic variables are defined to be the sum of a static value and a small fluctuation. Fundamental assumptions are then made about the magnitude of the acoustic fluctuation, the magnitude of the transport coefficients, and the amount of dispersion caused by the relaxation mechanism. The system for ranking the terms in the basic equations is then presented.

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a Taylor series,

$$\mu_B = (\mu_B)_0 + \left(\left.\frac{\partial \mu_B}{\partial P}\right|_{s,q}\right)_0 (P - P_0) + \left(\left.\frac{\partial \mu_B}{\partial s}\right|_{P,q}\right)_0 (s - s_0) + \left(\left.\frac{\partial \mu_B}{\partial q}\right|_{P,s}\right)_0 (q - q_0) + \cdots \quad .$$

Use of the ranking scheme indicates that the only term in $\nabla(\mu_B\nabla\cdot\mathbf{u})$ that is not negligible is $(\mu_B)_0\nabla\cdot\mathbf{u}$. Thus the ranking system will show us that the variation in the transport coefficients is a negligible effect, but it is simpler for us to treat them as constants from this point on.

We may define each thermodynamic variable to be the sum of a static value, which is denoted by the subscript 0, and a small fluctuation, which is denoted by the superscript $'$,

$$\rho \equiv \rho_0 + \rho' \quad , \tag{2.46}$$

$$P \equiv P_0 + p' \quad , \tag{2.47}$$

$$T \equiv T_0 + T' \quad , \tag{2.48}$$

$$s \equiv s_0 + s' \quad , \tag{2.49}$$

$$\mathbf{u} \equiv \mathbf{u}_0 + \mathbf{u}' \quad , \tag{2.50}$$

$$q \equiv q_0 + q' \quad , \tag{2.51}$$

$$q^* \equiv q_0^* + q^{*\prime} \quad . \tag{2.52}$$

Note that because the fluid was assumed to be initially quiet, the static value of the particle velocity is zero, that is, $|\mathbf{u}_0| = 0$. We therefore drop the unnecessary prime from $\mathbf{u}'$ and refer to the acoustic fluctuation in the particle velocity simply as $\mathbf{u}$. Recall that the static values of q and q^* are the same (i.e., $q_0 = q_0^*$). Equations (2.46) through (2.52) may be substituted directly in the continuity, momentum, entropy, and rate equations, but this is deferred until after the ranking system is presented.

It is well known that for most fluids the linear, lossless, wave equation (the *classical* wave equation) is a good model for describing the behaviour of small-signal acoustic waves. The classical wave equation may be obtained by retaining only linear, lossless terms in the basic equations. A linear, lossless term is a linear term with a coefficient that does not involve a *transport coefficient* or the derivative of the equilibrium variable q^* with respect to another thermodynamic variable. Examples are $\partial\rho'/\partial t$ and $\rho_0(\nabla\cdot\mathbf{u})$. The linear, lossless terms are therefore ranked as first-order terms.

On the other hand, if one wants a better approximation — an approximation that accounts for the weak effects of viscosity, heat conduction, relaxation, and nonlinearity — then more terms must be retained. The terms to be retained are linear relaxation terms, linear dissipation terms, and quadratic nonlinearity terms. These terms, which are defined below, are ranked as second-order terms and represent the most important effects of viscosity, heat conduction, relaxation, and nonlinearity. A linear relaxation term is a linear term with a coefficient that involves a derivative of q^* but not a transport coefficient. An example is $\rho'\left(\left.\frac{\partial q^*}{\partial \rho}\right|_s\right)_0$. A linear dissipation term is a linear term with a coefficient

that contains a transport coefficient but not a derivative of q^*. For example, $\mu\nabla^2\mathbf{u}$ is a linear dissipation term. Terms containing a quadratic nonlinear term with a coefficient that does not contain either a transport coefficient or a derivative of q^* are referred to as quadratic nonlinearity terms; examples are $\rho'(\partial\mathbf{u}/\partial t)$ and $\rho'(\nabla\cdot\mathbf{u})$. The names of the various terms are reflective of their physical interpretation.

Since the effects of viscosity, heat conduction, relaxation, and nonlinearity are assumed small, any term representing the interaction of any two second-order effects would be expected to be negligible. Such terms are encompassed in the third major category, higher-order terms. An example of a higher-order term is the cubic term, $\rho'(\mathbf{u}\cdot\nabla)\mathbf{u}$.

To quantify the ideas in the previous paragraphs, we now state three fundamental assumptions about the magnitude of the various terms: First, the magnitude of the particle velocity $|\mathbf{u}|$ is assumed to be small with respect to the sound speed c_0, that is,

$$|\mathbf{u}| \ll c_0 \quad .$$

It turns out that this assumption also implies that the magnitude of the pressure fluctuation p' is small with respect to $\rho_0 c_0^2$, and that the magnitude of the density fluctuation ρ' is small with respect to ρ_0 (see Appendix B),

$$|p'| \ll \rho_0 c_0^2 \quad ,$$

$$|\rho'| \ll \rho_0 \quad .$$

To aid in the subsequent ranking, we state that the nondimensional acoustic fluctuations $p'/\rho_0 c_0^2$, ρ'/ρ_0, and $\mathbf{u}/c_0$ are of order ε, where ε is a small parameter. Our second fundamental assumption is that for the highest frequency of interest the effects of shear viscosity, bulk viscosity, and heat conduction are small. This means that, when suitably nondimensionalized (see Appendix B), the transport coefficients μ, μ_B, and κ are all small quantities of order υ, where υ is a small parameter. Our third fundamental assumption is that the dispersion caused by the relaxation is of order $\mathcal{M}$, where $\mathcal{M}$ is a small nondimensional parameter that is formally defined later. At this point it is appropriate to recall that two fundamental assumptions were previously made: the deviation from equilibrium is assumed to be small enough that the thermodynamic fluxes and forces may be linearly related, and the cross-effects are assumed negligible.

Use of the ranking system makes it relatively easy to deal with special cases. For the case of small-signal wave motion in a lossless fluid, only terms of order ε need to be retained. For finite-amplitude wave motion in a lossless

fluid, terms of order ε and ε^2 must be retained. For finite-amplitude signals in a homogeneous, thermoviscous, relaxing fluid, terms of order ε, ε^2, $\mathcal{M}\varepsilon$, $\upsilon\varepsilon$ must all be retained. Thus, the basic equations can be simplified and, in most cases, combined to form a wave equation corresponding to the situation at hand.

It is important to note that not all second-order nonlinearity terms generate similar effects. The effects of nonlinearity may be roughly divided into two catagories: cumulative effects, that is, effects which grow with distance, and noncumulative or *local* effects. Since no *a priori* method exists to indicate whether a particular second-order nonlinearity term will result in effects that grow with distance, all second-order terms must be retained. Under some circumstances local effects can be important because, although they do not grow with distance, they can propagate. Thus, a local effect generated at one location may influence (or render impossible) the measurement of a local effect at a second location.

If Eqs. (2.46) through (2.52) are substituted into the continuity, momentum, entropy, and rate equations as well as the Taylor series expansions of the state equations and if the terms that are of higher order are neglected, the following equations are obtained:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\mathbf{u} = -\rho'\,\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\rho' \quad , \tag{2.53}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = -\rho' \frac{\partial \mathbf{u}}{\partial t} - \rho_0(\mathbf{u}\cdot\nabla)\mathbf{u} + \mu\nabla^2\mathbf{u} + \left(\mu_B + \frac{\mu}{3}\right)\nabla(\nabla\cdot\mathbf{u}) \quad , \tag{2.54}$$

$$\rho_0 T_0 \frac{\partial s'}{\partial t} = \kappa\nabla^2 T' \tag{2.55}$$

$$\frac{\partial q'}{\partial t} + \mathbf{u}\cdot\nabla q' = \frac{q' - q^{*\prime}}{\tau_0} \quad , \tag{2.56}$$

$$p' = \rho'\left(\left.\frac{\partial P}{\partial \rho}\right|_{s,q}\right)_0 + s'\left(\left.\frac{\partial P}{\partial s}\right|_{\rho,q}\right)_0 + q'\left(\left.\frac{\partial P}{\partial q}\right|_{\rho,s}\right)_0 + \tfrac{1}{2}(\rho')^2\left(\left.\frac{\partial^2 P}{\partial \rho^2}\right|_{s,q}\right)_0 \quad , \tag{2.57}$$

$$T' = \rho'\left(\left.\frac{\partial T}{\partial \rho}\right|_{s,q}\right)_0 + s'\left(\left.\frac{\partial T}{\partial s}\right|_{\rho,q}\right)_0 + q'\left(\left.\frac{\partial T}{\partial q}\right|_{\rho,s}\right)_0 + \tfrac{1}{2}(\rho')^2\left(\left.\frac{\partial^2 T}{\partial \rho^2}\right|_{s,q}\right)_0 \quad . \tag{2.58}$$

It turns out that both the momentum equation and the rate equation may be further simplified, but this is deferred until later.

2–3 The Sound Speed in a Relaxing Fluid

The defining relation for the sound speed is usually written as

$$c^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_s \quad . \tag{2.59}$$

Since three independent thermodynamic variables are required to completely specify the state of a fluid with a single relaxation mechanism and since Eq. (2.59) only specifies how two of them are to be used, the sound speed for a relaxing fluid may be defined in a multitude of ways. It turns out (as we see in the next chapter) that the following two defining relations play an important role in acoustic wave propagation in a relaxing fluid:

$$(c^\infty)^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_{s,q} = \frac{\partial}{\partial \rho} P(\rho, s, q) \quad , \tag{2.60}$$

$$(c^0)^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_{s,q=q^*} = \frac{\partial}{\partial \rho}\{P[\rho, s, q^*(\rho, s)]\} \quad . \tag{2.61}$$

Equation (2.60) is the defining relation for the frozen sound speed c^∞, the sound speed for a fixed (*frozen*) value of q that is not necessarily the equilibrium value q^*. Equation (2.61) is the defining relation for the equilibrium sound speed c^0, the sound speed for the equilibrium value of $q = q^*$ that is not necessarily a constant. It turns out (as we see in the next chapter) that the frozen sound speed is the appropriate sound speed for a fluid with a very long relaxation time and/or for a very high frequency signal. In this case, the signal disturbs a fluid particle at a rate that is much faster than the rate of the relaxation mechanism, that is, the fluid is effectively frozen in a single state. Thus, q is constant. On the other hand, the equilibrium sound speed is the appropriate sound speed for a fluid with a very short relaxation time and/or for a very low frequency signal. In this case, the signal disturbs a fluid particle at a rate that is much slower than the rate of the relaxation mechanism. The advancement variable q is thus always equal to its equilibrium value q^*, even though q^* changes with the passing acoustic wave.[6]

When evaluated at static conditions, the values of the frozen and equilibrium sound speeds are denoted with a subscript 0,

$$(c_0^\infty)^2 \equiv \left(\left.\frac{\partial P}{\partial \rho}\right|_{s,q}\right)_0 \quad , \tag{2.62}$$

[6] For a more detailed explanation of the roles of the frozen and equilibrium sound speeds, see, for example, Vincenti and Kruger (1965, Chap. VIII, Secs. 3 and 4).

$$(c_0^0)^2 \equiv \left(\frac{\partial P}{\partial \rho}\bigg|_{s,q=q^*} \right)_0 \quad . \tag{2.63}$$

Magnitude of the difference between the two sound speeds

When the ranking system was presented, we assumed that the dispersion caused by the relaxation mechanism was a small quantity of order $\mathcal{M}$. An expression for the nondimensional difference between the equilibrium and frozen sound speeds, which is denoted m and was previously assumed small, may be obtained in the following manner: The total differential of the pressure is first obtained and is then specialized for equilibrium. The difference between the frozen and equilibrium sound speeds follows directly. The total differential of the pressure is

$$dP = \frac{\partial P}{\partial \rho}\bigg|_{s,q} d\rho + \frac{\partial P}{\partial s}\bigg|_{\rho,q} ds + \frac{\partial P}{\partial q}\bigg|_{\rho,s} dq \quad . \tag{2.64}$$

However, at equilibrium the change in the entropy is zero, and the advancement variable q takes on its equilibrium value q^*,

$$dP = \frac{\partial P}{\partial \rho}\bigg|_{s,q} d\rho + \frac{\partial P}{\partial q}\bigg|_{\rho,s} dq^* \quad . \tag{2.65}$$

Rearranging using the defining relations for the equilibrium and frozen sound speeds and noting the following identity from calculus,[7]

$$\frac{\partial P}{\partial \rho}\bigg|_{q,s} \frac{\partial \rho}{\partial q}\bigg|_{P,s} \frac{\partial q}{\partial P}\bigg|_{\rho,s} = -1 \quad , \tag{2.66}$$

we obtain

$$(c^0)^2 = (c^\infty)^2 (1 - m) \quad , \tag{2.67}$$

where m is given by

$$m = \frac{\partial \rho}{\partial q}\bigg|_{P,s} \frac{\partial q^*}{\partial \rho}\bigg|_{s} \quad . \tag{2.68}$$

[7] For a general proof of the identity

$$\frac{\partial x}{\partial y}\bigg|_z \frac{\partial y}{\partial z}\bigg|_x \frac{\partial z}{\partial x}\bigg|_y = -1 \quad ,$$

see, for example, Van Wylen and Sonntag (1976, p. 366).

Rearranging Eq. (2.67) yields a more readily interpreted definition for m,

$$\frac{(c^\infty)^2 - (c^0)^2}{(c^\infty)^2} = m \quad . \tag{2.69}$$

The magnitude of m was assumed to be of order $\mathcal{M}$, where $\mathcal{M}$ is a small quantity. As with the frozen and equilibrium sound speeds, m has a static value that is denoted m_0.

As the value of m tends to 0, the effects of relaxation become small. At the same time, the values of q and q^* tend to the same constant, equilibrium value, $q_0 = q_0^*$. In this limiting case, the equilibrium and frozen sound speeds are equal, and, since q is no longer a variable, we recover the usual definition of the sound speed as given in Eq. (2.59),

$$m \to 0 \qquad c^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_s = (c^\infty)^2 = (c^0)^2 \quad . \tag{2.70}$$

We denote the limiting case of $m \to 0$ in our notation by dropping the specification on q from the partial differential. This notation is also used to denote when it is inappropriate to distinguish between the equilibrium and frozen forms of a thermodynamic fluid property; see below. When evaluated at *static conditions*, the sound speed for the $m = 0$ case is referred to as the small-signal sound speed and is denoted c_0,

$$c_0^2 \equiv \left(\left.\frac{\partial P}{\partial \rho}\right|_s\right)_0 \quad . \tag{2.71}$$

To show that the difference between the equilibrium and frozen forms of a first-order term is a second-order term, we may rearrange the definition of m and then multiply through by a first-order term, say, the fluctuation in the density,

$$(c_0^\infty)^2\rho' = (c_0^0)^2\rho' + m_0(c_0^0)^2\rho' \quad .$$

Since the term $m_0(c_0^0)^2\rho'$ is of second order, it is not appropriate to distinguish between the equilibrium and frozen forms of the sound speed. (A higher-order term would be introduced.) The above expression is thus written correct to second order as

$$(c_0^\infty)^2\rho' = (c_0^0)^2\rho' + m_0(c_0)^2\rho' \quad . \tag{2.72}$$

We may generalize the foregoing discussion. Since the difference between the equilibrium and frozen forms of a thermodynamic fluid property is a second-order term, it is appropriate to distinguish between the equilibrium and frozen

forms of a thermodynamic fluid property *only when the property is a coefficient of a first-order term.* This is because the difference between the equilibrium and frozen forms of a first-order term is a second-order term (of order $\mathcal{M}\varepsilon$), whereas the difference between the equilibrium and frozen forms of a second-order term is a higher-order term. In second-order terms, we therefore neglect the effects of relaxation. This is indicated notationally by not specifying what is to be done with q in the partial derivatives that are the defining relations for the thermodynamic fluid properties.[8]

2–4 First-Order Acoustics: Small Signals in a Lossless Fluid

In this section the case of small-signal wave motion in a lossless fluid is considered. The terms in the basic equations that pertain to nonlinearity, viscosity, heat conduction, and relaxation are dropped, and it is then seen that the basic equations for this case are composed only of first-order terms. It is also noted that neglecting relaxation and heat conduction eliminates the need for s and q as variables. The motivation for developing some first-order relations at this time is to assist us in simplifying the second-order relations later. First-order relations are of assistance because the dependent variables in second-order terms may be replaced using first-order relations without changing the overall level of approximation; see, for example, Lighthill (1956).

If all second-order terms are neglected, the following first-order forms of the continuity and momentum equations, Eqs. (2.53) and (2.54), are obtained:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \quad , \tag{2.73}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = 0 \quad . \tag{2.74}$$

Equations (2.73) and (2.74) are referred to as the linear continuity and momentum equations.

We now discuss the rate and entropy equations or, more accurately, the lack of need for them. Since the fluid is assumed lossless, the fluid is assumed

[8] If the difference between the frozen and equilibrium forms of second-order terms were not negligible, that is to say, if m were not small, then frozen and equilibrium forms of both the loss coefficient (defined below) and the parameter of nonlinearity (also defined below) would emerge. However, in this work, m is assumed to be a small parameter, and the differences between the equilibrium and frozen forms of terms containing the loss coefficient and the parameter of nonlinearity are of higher order.

to be in a state of thermodynamic equilibrium at all times. The variation of s and q for any fluid particle is thus zero. Moreover, because the fluid is assumed homogeneous, both s and q are constant throughout the fluid. The rate and entropy equations thus reduce to

$$q = q^* = \text{constant} \quad ,$$

$$s = \text{constant} \quad .$$

Because s and q are constant throughout the fluid, the state equation, Eq. (2.4), reduces to

$$P = P(\rho, s = \text{constant}, q = \text{constant}) \quad . \tag{2.75}$$

Equation (2.75) is sometimes referred to as the isentropic equation of state. The linearized Taylor series expansion of Eq. (2.75) is

$$p' = \left(\left. \frac{\partial P}{\partial \rho} \right|_{s,q} \right)_0 \rho' \quad .$$

However, as noted above, it is inappropriate to distinguish between the equilibrium and frozen forms of the sound speed when relaxation is not important. The linearized Taylor series expansion of the state equation for a lossless fluid is therefore written as

$$p' = c_0^2 \rho' \quad . \tag{2.76}$$

Equations (2.73), (2.74), and (2.76) are the first-order forms of the continuity, momentum, and state equations and are valid for small-signal wave motion in a lossless fluid. These equations may be combined to form wave equations in p' and ρ',

$$\nabla^2 p' = \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} \quad , \tag{2.77}$$

$$\nabla^2 \rho' = \frac{1}{c_0^2} \frac{\partial^2 \rho'}{\partial t^2} \quad . \tag{2.78}$$

2–5 Second-Order Forms of the Basic Equations

In this section a consistent second-order form of the momentum equation is obtained. The order ε^2 terms in both the continuity and momentum equations are then expressed in terms of a new variable called the Lagrangian density $\mathcal{L}$.

The Lagrangian density is related to some of the *local* finite-amplitude effects. Next, the second-order forms of the entropy, state, and rate equations are obtained. These equations are then combined into a single equation that is referred to as the modified state equation. Last, the integral of the modified state equation is obtained.

Continuity and momentum equations

No terms in the expanded continuity equation, Eq. (2.53), are of higher order. Thus, the second-order form of the continuity equation is exact. Note that the second-order terms in the continuity equation are quadratic nonlinearity terms and need only be retained when finite-amplitude effects are of interest.

A consistent second-order approximation of the expanded momentum equation, Eq. (2.54), is now obtained for a signal propagating far from boundaries in an initially irrotational fluid. We first introduce the vorticity $\boldsymbol{\Omega}$,

$$\boldsymbol{\Omega} \equiv \boldsymbol{\nabla}\times\mathbf{u} \quad . \tag{2.79}$$

The terms $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ and $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})$ in the momentum equation may be expanded using vector identities [Gradshteyn and Ryzhik 1980, Eq. (10.31.3)] and the definition of the vorticity to give

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \tfrac{1}{2}\boldsymbol{\nabla}(u^2) + \boldsymbol{\Omega}\times\mathbf{u}$$

and

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) = \nabla^2\mathbf{u} + \boldsymbol{\nabla}\times\boldsymbol{\Omega} \quad .$$

Inserting these relations into Eq. (2.54) yields

$$\begin{aligned}\rho_0\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\nabla}p' = {} & \left(\mu_B + \tfrac{4}{3}\mu\right)\nabla^2\mathbf{u} - \tfrac{1}{2}\rho_0\boldsymbol{\nabla}u^2 - \rho'\frac{\partial\mathbf{u}}{\partial t} \\ & -\rho_0(\boldsymbol{\Omega}\times\mathbf{u}) + \left(\mu_B + \frac{\mu}{3}\right)\boldsymbol{\nabla}\times\boldsymbol{\Omega} \quad ,\end{aligned}$$

where $u^2 \equiv \mathbf{u}\cdot\mathbf{u}$. In Appendix B, however, it is shown that for a signal propagating far from any boundaries in an initially irrotational fluid, the vorticity is a higher-order term. The vorticity terms in the above relation are therefore neglected.[9] The remaining terms form the second-order approximation of the momentum equation,

$$\rho_0\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\nabla}p' = \rho_0\nu\mathrm{V}\nabla^2\mathbf{u} - \rho'\frac{\partial\mathbf{u}}{\partial t} - \tfrac{1}{2}\rho_0\boldsymbol{\nabla}u^2 \quad , \tag{2.80}$$

[9] The interested reader is referred to Appendix B where vorticity and the irrotational flow assumption are discussed at length.

where ν is the kinematic viscosity and V is is a dimensionless number called the viscosity number,

$$\nu \equiv \frac{\mu}{\rho_0} \quad , \tag{2.81}$$

$$\mathrm{V} \equiv \frac{4}{3} + \frac{\mu_{\mathrm{B}}}{\mu} \quad . \tag{2.82}$$

The viscosity number indicates the relative importance of the bulk viscosity to the shear viscosity. Estimates of the viscosity number and the other nondimensional numbers that appear are given in Appendix B.

A quantity that aids in the interpretation of local nonlinear effects is the Lagrangian density $\mathcal{L}$,

$$\mathcal{L} \equiv \frac{\rho_0}{2}u^2 - \frac{(p')^2}{2\rho_0 c_0^2} \quad . \tag{2.83}$$

The two terms on the right-hand side are, respectively, the kinetic and potential energy densities. It turns out that terms that contain the Lagrangian density correspond to local nonlinear effects; see, for example, Aanonsen *et al.* (1984). The primary advantage of the term $\mathcal{L}$ is that it makes it easy to see what terms are to be dropped when the wave motion is progressive; see Hamilton and Blackstock (1988). The Lagrangian density and local effects are discussed in greater detail in later chapters.

We now introduce the Lagrangian density in the second-order forms of the continuity and momentum equations, Eqs. (2.53) and (2.80). Recall that first-order relations may be used to modify second-order terms. Thus the second-order terms in the continuity equation may be rewritten as

$$\rho' \nabla \cdot \mathbf{u} = -\frac{1}{2\rho_0 c_0^4}\frac{\partial (p')^2}{\partial t} \tag{2.84}$$

and

$$\mathbf{u} \cdot \nabla \rho' = -\frac{\rho_0}{2c_0^2}\frac{\partial u^2}{\partial t} \quad , \tag{2.85}$$

and the term $\rho'(\partial \mathbf{u}/\partial t)$ in the momentum equation may be rewritten as

$$\rho' \frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla (p')^2}{2\rho_0 c_0^2} \quad . \tag{2.86}$$

Use of Eqs. (2.84), (2.85), and (2.86) as well as the definition of the Lagrangian density in the continuity and momentum equations yields the desired equivalent

second-order forms of the continuity and momentum equations,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = \frac{1}{\rho_0 c_0^4} \frac{\partial (p')^2}{\partial t} + \frac{1}{c_0^2} \frac{\partial \mathcal{L}}{\partial t} \tag{2.87}$$

and

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = \rho_0 \nu \mathrm{V} \nabla^2 \mathbf{u} - \nabla \mathcal{L} \quad . \tag{2.88}$$

Entropy, state, and rate equations

The rank-ordering process leads to the linearized form of the expanded entropy equation, Eq. (2.55). Both terms in Eq. (2.55) are of second order. To eliminate T' from the equation, we use the first-order approximation of Eq. (2.58),

$$T' = \left(\left. \frac{\partial T}{\partial \rho} \right|_{s,q} \right)_0 \rho' \quad . \tag{2.89}$$

Substituting this expression into the entropy equation and then using the first-order wave equation in terms of the density, Eq. (2.78), leads to

$$\frac{\partial s'}{\partial t} = \frac{\kappa}{\rho_0 T_0 c_0^2} \left(\left. \frac{\partial T}{\partial \rho} \right|_{s,q} \right)_0 \frac{\partial^2 \rho'}{\partial t^2} \quad .$$

This relation may be integrated once noting that the integration constant must be zero to satisfy quiet conditions. Recalling our discussion on the order of the difference between the equilibrium and frozen forms of a thermodynamic fluid property, we see that the following is an equally valid second-order approximation of the entropy in a relaxing fluid:

$$s' = \frac{\kappa}{\rho_0 T_0 c_0^2} \left(\left. \frac{\partial T}{\partial \rho} \right|_{s} \right)_0 \frac{\partial \rho'}{\partial t} \quad . \tag{2.90}$$

Note that the effects of finite amplitude, viscosity, and relaxation do not, within the second-order approximation, contribute to the entropy production. Thus, the entropy equation need only be considered when heat conduction is of interest.

The ranking process also resulted in a simplified version of the Taylor series expansion of the state equation, Eq. (2.57). Recalling our earlier discussion on the order of the difference between the equilibrium and frozen forms of thermodynamic fluid properties, we may rewrite Eq. (2.57) as follows:

$$p' = (c_0^\infty)^2 \rho' + s' \left(\left. \frac{\partial P}{\partial s} \right|_{\rho} \right)_0 + q' \left(\left. \frac{\partial P}{\partial q} \right|_{\rho,s} \right)_0 + \frac{c_0^2}{\rho_0} \frac{B}{2A} (\rho')^2 \quad , \tag{2.91}$$

where B/A is referred to as the parameter of nonlinearity,

$$\frac{B}{A} = \frac{\rho_0}{c_0^2}\left(\left.\frac{\partial^2 P}{\partial \rho^2}\right|_s\right)_0 \quad . \tag{2.92}$$

The defining relation for the frozen sound speed, Eq. (2.60), was used in obtaining Eq. (2.91). The second-order terms in Eq. (2.91) must be included when the effects of heat conduction, relaxation, or finite-amplitude, respectively, are of interest.

A consistent second-order form of the expanded rate equation, Eq. (2.56), is now obtained. To do such, we require some information about the order of q'. The linearized Taylor series expansion of q' is

$$q'(p', \rho', s') = p'\left(\left.\frac{\partial q}{\partial P}\right|_{\rho,s}\right)_0 + \rho'\left(\left.\frac{\partial q}{\partial \rho}\right|_{P,s}\right)_0 + s'\left(\left.\frac{\partial q}{\partial s}\right|_{P,\rho}\right)_0 \quad .$$

However, use of Eq. (2.66) in the expansion for q' leads to

$$q'(p', \rho', s') = \left(\left.\frac{\partial q}{\partial \rho}\right|_{P,s}\right)_0 \left(\rho' - p'/(c_0^\infty)^2\right) + s'\left(\left.\frac{\partial q}{\partial s}\right|_{P,\rho}\right)_0 \quad .$$

Noting Eq. (2.72), we see that the first term on the left-hand side of the above equation is of second order. Since s' is also of second order, the quantity q' must be of second order. Thus the convective term in the rate equation must be of higher order, and a consistent second-order approximation of the rate equation is

$$\frac{\partial q'}{\partial t} = \frac{q^{*\prime} - q'}{\tau_0} \quad . \tag{2.93}$$

The rate equation must be included when relaxation effects are of interest.

Combining the entropy, state, and rate equations

The second-order approximations of the entropy, state, and rate equations may be combined into a single equation that is referred to as the modified state equation. Our starting point is the time derivative of the rate equation, Eq. (2.93):

$$\tau_0 \frac{\partial}{\partial t}\left(\frac{\partial q'}{\partial t}\right) + \frac{\partial}{\partial t}(q' - q^{*\prime}) = 0 \quad . \tag{2.94}$$

Our procedure is as follows: The entropy equation is substituted into the state equation with the help of some thermodynamic identities from Appendix A. The

resulting equation is then rearranged to yield an expression for $\partial q'/\partial t$. This expression and an expression for $\partial q^{*\prime}/\partial t$ are then substituted into Eq. (2.94). This yields the modified state equation. An equivalent form of the modified state equation is then obtained by integration.

The entropy and state equations, Eqs. (2.90) and (2.91), respectively, are now combined via the term $\left(\frac{\partial P}{\partial s}\Big|_\rho\right)_0 \frac{\partial s'}{\partial t}$ and then rearranged to yield an expression for $\partial q'/\partial t$. Use of the static form of Eq. (A.11) from Appendix A and the time derivative of the entropy equation yields

$$\left(\frac{\partial P}{\partial s}\Big|_\rho\right)_0 \frac{\partial s'}{\partial t} = \frac{\kappa\,\rho_0}{T_0 c_0^2}\left(\frac{\partial T}{\partial \rho}\Big|_s\right)_0^2 \frac{\partial^2 \rho'}{\partial t^2} \quad .$$

However, noting the definition of the thermal expansion coefficient, Eq. (A.22), and using the static form of a thermodynamic identity from Appendix A, Eq. (A.26), we see that

$$\left(\frac{\partial P}{\partial s}\Big|_\rho\right)_0 \frac{\partial s'}{\partial t} = \frac{\kappa}{\rho_0}\left(\frac{1}{c_{v_0}} - \frac{1}{c_{p_0}}\right)\frac{\partial^2 \rho'}{\partial t^2} \quad , \tag{2.95}$$

where c_{v_0} and c_{p_0} are the specific heats at constant volume and pressure, respectively. Equation (2.95) can be rearranged and placed in the following form:

$$\left(\frac{\partial P}{\partial s}\Big|_\rho\right)_0 \frac{\partial s'}{\partial t} = \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial^2 \rho'}{\partial t^2} \quad , \tag{2.96}$$

where γ is the ratio of specific heats and Pr is the Prandtl number,

$$\gamma \equiv \frac{c_{p_0}}{c_{v_0}} \quad , \tag{2.97}$$

$$\mathrm{Pr} \equiv \frac{\mu c_{p_0}}{\kappa} \quad . \tag{2.98}$$

The Prandtl number indicates the relative importance of viscosity to heat conduction. Estimates of the Prandtl number are given in Appendix B. Substituting Eq. (2.96) into the time derivative of the state equation, Eq. (2.91), and rearranging leads to an expression for $\frac{\partial q'}{\partial t}$,

$$\left(\frac{\partial P}{\partial q}\Big|_{\rho,s}\right)_0 \frac{\partial q'}{\partial t} = \frac{\partial p'}{\partial t} - (c_0^\infty)^2\frac{\partial \rho'}{\partial t} - \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial^2 \rho'}{\partial t^2} - \frac{c_0^2}{\rho_0}\frac{B}{A}\rho'\frac{\partial \rho'}{\partial t} \quad . \tag{2.99}$$

An expression for $\frac{\partial q^{*\prime}}{\partial t}$ is obtained from the time derivative of the linearized Taylor series expansion of $q^{*\prime}$,

$$\frac{\partial q^{*\prime}}{\partial t} = \left(\left.\frac{\partial q^*}{\partial \rho}\right|_s \right)_0 \frac{\partial \rho'}{\partial t} + \left(\left.\frac{\partial q^*}{\partial s}\right|_\rho \right)_0 \frac{\partial s'}{\partial t} \quad .$$

The second-order term on the right-hand side drops out because the change in the entropy at equilibrium is zero. Thus, the above relation simplifies to

$$\frac{\partial q^{*\prime}}{\partial t} = \left(\left.\frac{\partial q^*}{\partial \rho}\right|_s \right)_0 \frac{\partial \rho'}{\partial t} \quad . \tag{2.100}$$

An expression for $\frac{\partial}{\partial t}(q' - q^{*\prime})$ may be obtained by combining Eqs. (2.100) and (2.99),

$$\left(\left.\frac{\partial P}{\partial q}\right|_{\rho,s} \right)_0 \frac{\partial}{\partial t}(q' - q^{*\prime}) = \frac{\partial p'}{\partial t} - \left[(c_0^\infty)^2 + \left(\left.\frac{\partial P}{\partial q}\right|_{\rho,s} \right)_0 \left(\left.\frac{\partial q^*}{\partial \rho}\right|_s \right)_0 \right] \frac{\partial \rho'}{\partial t}$$
$$- \frac{\nu(\gamma - 1)}{\mathrm{Pr}} \frac{\partial^2 \rho'}{\partial t^2} - \frac{c_0^2}{\rho_0} \frac{B}{A} \rho' \frac{\partial \rho'}{\partial t} \quad .$$

However, using Eqs. (2.66), (2.67), and (2.68), we see that the coefficient of the second term on the right-hand side is simply the equilibrium sound speed. Thus, we obtain

$$\left(\left.\frac{\partial P}{\partial q}\right|_{\rho,s} \right)_0 \frac{\partial}{\partial t}(q' - q^{*\prime}) = \frac{\partial p'}{\partial t} - (c_0^0)^2 \frac{\partial \rho'}{\partial t} - \frac{\nu(\gamma - 1)}{\mathrm{Pr}} \frac{\partial^2 \rho'}{\partial t^2} - \frac{c_0^2}{\rho_0} \frac{B}{A} \rho' \frac{\partial \rho'}{\partial t} \quad . \tag{2.101}$$

The modified state equation is now formed. Substitution of Eqs. (2.99) and (2.101) into Eq. (2.94) yields an expression that may be integrated once with respect to time. The integration constant must be zero in order to satisfy quiet conditions. The result is

$$\tau_0 \frac{\partial}{\partial t} \left(p' - (c_0^\infty)^2 p' - \frac{\nu(\gamma - 1)}{\mathrm{Pr}} \frac{\partial \rho'}{\partial t} - \frac{c_0^2}{\rho_0} \frac{B}{2A} (\rho')^2 \right)$$
$$+ \left(p' - (c_0^0)^2 \rho' - \frac{\nu(\gamma - 1)}{\mathrm{Pr}} \frac{\partial \rho'}{\partial t} - \frac{c_0^2}{\rho_0} \frac{B}{2A} (\rho')^2 \right) = 0 \quad . \tag{2.102}$$

Equation (2.102) is the modified state equation, a combination of the entropy, state, and rate equations. The basis for our earlier comments about the relationship between the equilibrium and frozen sound speeds and the relaxation

time/signal frequency is now apparent. If the relaxation time is very large, Eq. (2.102) simplifies into an expression that depends only on the frozen sound speed. Conversely, for a small relaxation time, only the equilibrium sound speed appears.

Equation (2.102) may be integrated by following the procedure of Rudenko and Soluyan (1977, p. 83). Use of Eq. (2.69) leads to the following:

$$\tau_0 \frac{\partial}{\partial t}\left(p' - (c_0^0)^2\rho' - \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial \rho'}{\partial t} - \frac{c_0^2}{\rho_0}\frac{B}{2A}(\rho')^2\right)$$
$$+ \left(p' - (c_0^0)^2\rho' - \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial \rho'}{\partial t} - \frac{c_0^2}{\rho_0}\frac{B}{2A}(\rho')^2\right) - m_0\,\tau_0\,c_0^2\frac{\partial \rho'}{\partial t} = 0 \quad .$$

The above equation is in the form of a linear first-order differential equation that can be solved by means of an integrating factor [see, for example, Kreider *et al.* (1966, p. 97)]. The result is

$$p' = (c_0^0)^2\rho' + \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial \rho'}{\partial t} + \frac{c_0^2}{\rho_0}\frac{B}{2A}(\rho')^2 + m_0\,c_0^2\int_{-\infty}^{t}\frac{\partial \rho'}{\partial t}e^{-(t-y)/\tau_0}\,dy \quad . \tag{2.103}$$

The integral in Eq. (2.103) may be integrated by parts, and the resulting expression is

$$p' = (c_0^\infty)^2\rho' + \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial \rho'}{\partial t} + \frac{c_0^2}{\rho_0}\frac{B}{2A}(\rho')^2 - \frac{m_0\,c_0^2}{\tau_0}\int_{-\infty}^{t}\rho' e^{-(t-y)/\tau_0}\,dy \quad . \tag{2.104}$$

Equation (2.104) is referred to as the integral of the modified state equation. Equation (2.104) corresponds to Eq. (IV–1.20) in Rudenko and Soluyan's book (1977) except that Eq. (2.104) also accounts for heat conduction. The second, third, and fourth terms on the right-hand side of the Eq. (2.104) are associated with heat conduction, finite-amplitude effects, and relaxation, respectively.

For the special case of $\omega\tau_0 \ll 1$ it is difficult to distinguish between other loss terms and the leading-order effects of relaxation. Integrating Eq. (2.103) by parts and neglecting terms of order $O[\epsilon(\omega\tau_0)^2]$ and higher yields

$$p' = (c_0^0)^2\rho' + \left(\frac{\nu(\gamma-1)}{\mathrm{Pr}} + m_0 c_0^2\tau_0\right)\frac{\partial \rho'}{\partial t} + \frac{c_0^2}{\rho_0}\frac{B}{2A}(\rho')^2 \quad . \tag{2.105}$$

In particular, the leading order effects of relaxation are difficult to distinguish from bulk viscosity because it is, apparently, difficult to devise a measurement technique which isolates the bulk viscosity from the relaxation.

Summary of second-order results

In this section, consistent second-order approximations of the continuity, momentum, entropy, state, and rate equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism were obtained. The equations are, respectively, Eqs. (2.53), (2.80), (2.90), (2.91), and (2.93). The Lagrangian density was introduced into the continuity and momentum equations, and equivalent forms of the equations, Eqs. (2.87) and (2.88), were obtained. The entropy, state, and rate equations were combined to form a new equation, which is called the modified state equation. Two equivalent forms of the modified state equation, Eqs. (2.102) and (2.104), were obtained.

2–6 Small-Signal Wave Motion in a Homogeneous, Thermoviscous, Relaxing Fluid

In this section the basic equations are simplified for the case of small-signal propagation in a homogeneous, thermoviscous, relaxing fluid. Rather than re-deriving the relations from the basic equations, we merely drop the quadratic nonlinearity terms from the second-order forms of the basic equations.[10] Since no quadratic nonlinearity terms exist in either the entropy or rate equations, Eqs. (2.90) and (2.93), respectively, they remain unchanged. Neglecting quadratic nonlinearity terms in the continuity yields the same equation that was obtained for small-signal propagation in a lossless fluid, Eq. (2.73). Neglecting quadratic nonlinearity terms in the momentum and state equations leads to the following:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla} p' = \rho_0 \nu \mathrm{V} \nabla^2 \mathbf{u} \tag{2.106}$$

and

$$p' = (c_0^\infty)^2 \rho' + s' \left(\left. \frac{\partial P}{\partial s} \right|_\rho \right)_0 + q' \left(\left. \frac{\partial P}{\partial q} \right|_{\rho,s} \right)_0 \quad . \tag{2.107}$$

[10]If one wanted to derive approximate forms of the basic equations valid for small-signal propagation in a homogeneous, thermoviscous fluid with a single relaxation mechanism starting from the basic equations, the procedure is identical to that used for finite-amplitude signals except that nonlinear terms may be neglected from the outset. It turns out, however, the effort saved in neglecting nonlinearity from the outset is small.

Dropping the quadratic nonlinearity term from the modified state equations, Eqs. (2.102) and (2.104), yields

$$\tau_0 \frac{\partial}{\partial t}\left(p' - (c_0^\infty)^2\rho' - \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial\rho'}{\partial t}\right) + \left(p' - (c_0^0)^2\rho' - \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial\rho'}{\partial t}\right) = 0 \tag{2.108}$$

and

$$p' = (c_0^\infty)^2\rho' + \frac{\nu(\gamma-1)}{\mathrm{Pr}}\frac{\partial\rho'}{\partial t} - \frac{m_0 c_0^2}{\tau_0}\int_{-\infty}^{t} \rho' e^{-(t-y)/\tau_0}\,dy \quad . \tag{2.109}$$

The forms of the continuity, momentum, entropy, state, and rate equations that are appropriate for small-signal propagation in a homogeneous, thermoviscous fluid with a single relaxation mechanism are, respectively, Eqs. (2.73), (2.106), (2.90), (2.107), and (2.93). The appropriate forms of the modified state equation are Eqs. (2.108) and (2.109).

2–7 Epilogue

In this chapter, the basic equations for a homogeneous thermoviscous fluid with a single relaxation mechanism were presented. The deviation from equilibrium was assumed to be small, and, accordingly, linear relations between the thermodynamic fluxes and forces were used. The possibility of a cross-effect between the bulk viscosity and relaxation was neglected. Assumptions were then made about the magnitude of the acoustic signal, the magnitude of the transport coefficients, and the amount of dispersion. A ranking system was introduced and used to simplify the basic equations into specialized forms for (1) small signals in lossless fluids, (2) small signals in homogeneous, thermoviscous, relaxing fluids, and (3) finite-amplitude signals in homogeneous, thermoviscous, relaxing fluids. Moreover, the Lagrangian density was introduced into the continuity and momentum equations, and the entropy, state, and rate equations were combined to form a new equation, the modified state equation.

CHAPTER 3

WAVE EQUATIONS FOR FINITE-AMPLITUDE SIGNALS IN A HOMOGENEOUS, THERMOVISCOUS FLUID WITH A SINGLE RELAXATION MECHANISM

3–1 Introduction

In this chapter second-order forms of the wave equation for finite-amplitude signals in a thermoviscous, relaxing fluid are developed. The wave equations are expressed in terms of the acoustic pressure p', the velocity potential ϕ, and two new variables—the modified acoustic pressure P' and the modified velocity potential Φ. The modified variables are defined in such a way that the Lagrangian density $\mathcal{L}$ does not appear in a wave equation that is expressed in terms of P' or Φ. Use of the modified pressure and modified velocity potential was developed by Naze Tjøtta and Tjøtta; see, for example, Aanonsen *et al.* (1984).

3–2 Wave Equation in Terms of the Acoustic Pressure p'

The wave equation in terms of the acoustic pressure p' is derived in this section. The procedure is as follows: The continuity and momentum equations containing the Lagrangian are combined to form one equation. This equation and the linear state equation are then used to eliminate the density fluctuation in the modified state equation. The result is the wave equation in terms of the acoustic pressure p'. After a modicum of rearrangement, the wave equation may be integrated and placed in an equally valid form that is referred to as the integral of the wave equation in terms of the acoustic pressure p'.

Combining the continuity and momentum equations, Eqs. (2.87) and (2.88), is a straightforward procedure: The time derivative of the continuity equation is subtracted from the divergence of the momentum equation. The viscosity term may, however, be rearranged so that it may later be readily combined with

the heat conduction term. Use of first-order forms of the continuity equation, the state equation, and the wave equation in p', Eqs. (2.73), (2.76), and (2.77), respectively, leads to the following first-order relation:

$$\nabla^2(\boldsymbol{\nabla}\cdot\mathbf{u}) = -\frac{1}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} \quad .$$

Use of the above yields the desired form of the combined continuity and momentum equation,

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p' + \frac{\nu \mathrm{V}}{c_0^4}\frac{\partial^3 p'}{\partial t^3} + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} + \frac{1}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2 \quad . \tag{3.1}$$

The wave equation in terms of the acoustic pressure is obtained by eliminating the density fluctuation ρ' in first and second-order terms (using, respectively, Eqs. (3.1) and the linear state equation, Eq. (2.76)) in the modified state equation, Eq. (2.102). The result is

$$\begin{aligned} \tau_0 \frac{\partial}{\partial t}\left[\nabla^2 p' - \frac{1}{(c_0^\infty)^2}\frac{\partial^2 p'}{\partial t^2} + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} + \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2\right] \\ + \left[\nabla^2 p' - \frac{1}{(c_0^0)^2}\frac{\partial^2 p'}{\partial t^2} + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} + \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2\right] = 0 \quad , \end{aligned} \tag{3.2}$$

where b/ρ_0 is the diffusivity of sound [Lighthill 1956, Eq. (19)] and β is the coefficient of nonlinearity,

$$b \equiv \rho_0 \nu \left(\mathrm{V} + \frac{\gamma - 1}{\mathrm{Pr}}\right) = \mu_\mathrm{B} + \frac{4}{3}\mu + \kappa\left(\frac{1}{c_v} + \frac{1}{c_p}\right) \quad , \tag{3.3}$$

$$\beta \equiv 1 + \frac{B}{2A} \quad . \tag{3.4}$$

Note that if the relaxation time τ_0 is very large, Eq. (3.2) simplifies to a wave equation that involves only the frozen sound speed. On the other hand, if the relaxation time is very short, the converse occurs; Eq. (3.2) simplifies to a wave equation that involves only the equilibrium sound speed. The third, fourth, and fifth terms in both major components of Eq. (3.2) represent, respectively, the effects of viscosity and heat conduction, local nonlinearity, and growing nonlinearity. Considered together, the first and second terms in both the major components represent linear, lossless wave motion at the frozen and equilibrium sound speeds, respectively.

Equation (3.2) may be integrated in the same way that the modified state equation, Eq. (2.102), was integrated. First, however, it must be rearranged using the following:

$$\frac{1}{(c_0^\infty)^2} = \frac{1}{(c_0^0)^2}(1 - m_0) \quad . \tag{3.5}$$

Substitution of Eq. (3.5) into Eq. (3.2) leads to

$$\begin{aligned} \tau_0 \frac{\partial}{\partial t}\left[\Box_0^2 p' + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} + \frac{\beta}{\rho_0 c_0^4}(p')^2\right] \\ + \left[\Box_0^2 p' + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} + \frac{\beta}{\rho_0 c_0^4}(p')^2\right] = -\frac{m_0 \tau_0}{c_0^2}\frac{\partial^3 p'}{\partial t^3} \quad , \end{aligned} \tag{3.6}$$

where $\Box^2$ is called the d'Alembertian operator and the subscript 0 indicates that this d'Alembertian operator uses the equilibrium sound speed,

$$\Box_0^2 \equiv \nabla^2 - \frac{1}{(c_0^0)^2}\frac{\partial^2}{\partial t^2} \quad .$$

As was the case with the modified state equation, Eq. (3.6) is in the form of a linear first-order differential equation that may be solved by way of an integrating factor (see, for example, Kreider *et al.* (1966, p. 97)). The result is

$$\begin{aligned} \Box_0^2 p' + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t} \frac{\partial^3 p'}{\partial y^3} e^{-(t-y)/\tau_0}\, dy = \\ -\left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} - \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2 \quad , \end{aligned} \tag{3.7}$$

which is here called the integral of the wave equation in terms of the acoustic pressure. The equation is a second-order approximation that consistently accounts for the effects of viscosity, heat conduction, relaxation, and nonlinearity. Each of the five terms in Eq. (3.7) represents a specific acoustic phenomenon. The three terms on the left-hand side represent, respectively, (1) small-signal wave motion in a lossless fluid, (2) the dissipation of the signal that is caused by viscosity and heat conduction, and (3) the effects of relaxation. The two terms on the right-hand side of the equation represent, respectively, (1) the local nonlinear effects and (2) the cumulative nonlinear effects. Clearly, if either thermoviscous or relaxation effects are negligible, the appropriate form of the wave equation is obtained by setting b or m, respectively, to zero. Similarly, if nonlinearity is

not of interest, then the appropriate form of the wave equation is obtained by neglecting the two terms on the right-hand side.

An equation that is equivalent to Eq. (3.7) may be obtained by combining Eq. (3.1) with the integral of the modified state equation, Eq. (2.104). This eliminates the need for the foregoing integration, but Eq. (3.2) would not be obtained.

As we noted in the previous chapter when examining the modified state equation, it is difficult to distinguish between other loss terms and the leading-order effects of relaxation for the special case of $\omega\tau_0 \ll 1$. Integrating the relaxation term in Eq. (3.7) by parts and neglecting terms of order $O[\epsilon(\omega\tau_0)^2]$ and higher yields

$$\Box_0^2 p' + \frac{(b + m_0\tau_0\rho_0 c_0^2)}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} = -\left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L} - \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2 \quad . \tag{3.8}$$

Note that the definition of the diffusivity b may be generalized to include the effects of relaxation in this special case of $\omega\tau_0 \ll 1$.

3–3 The Wave Equation in Terms of the Velocity Potential ϕ

The particle velocity may be expressed as the gradient of a scalar potential because the fluid is, within the second-order approximation, irrotational (see Appendix B),

$$\mathbf{u} = \boldsymbol{\nabla}\phi \quad . \tag{3.9}$$

The gradient of the velocity potential may therefore be used to replace the particle velocity in all equations. Moreover, first and second-order expressions for the pressure and the wave equation in terms of ϕ may be obtained. An expression for the Lagrangian $\mathcal{L}$ in terms of ϕ may also be obtained.

Before obtaining the second-order form of the wave equation in ϕ, we develop the other aforementioned relations. Integration of the first-order wave equation in terms of the particle velocity and setting the arbitrary static value of the potential ϕ_0 to zero leads to

$$\nabla^2\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = 0 \quad . \tag{3.10}$$

The second-order form of the momentum equation, Eq. (2.88), becomes (after integration)

$$p' = -\rho_0\frac{\partial\phi}{\partial t} + \rho_0\nu V\nabla^2\phi - \mathcal{L} \quad , \tag{3.11}$$

the first-order approximation of which is

$$p' = -\rho_0 \frac{\partial \phi}{\partial t} \quad . \tag{3.12}$$

The Lagrangian $\mathcal{L}$, which is defined in Eq. (2.83), may also be expressed in terms of ϕ,

$$\mathcal{L} = \frac{\rho_0}{2} \left[(\nabla \phi)^2 - \frac{1}{c_0^2} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] \quad . \tag{3.13}$$

Note, however, that the term $\Box^2 \phi^2$ expands identically as

$$\Box^2 \phi^2 = 2\phi \left(\nabla^2 \phi - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} \right) + 2 \left[(\nabla \phi)^2 - \frac{1}{c_0^2} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] \quad ,$$

where $\Box^2$ is the d'Alembertian operator using the small-signal sound speed,

$$\Box^2 \equiv \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \quad . \tag{3.14}$$

Noting Eq. (3.10), we see that, to second order, the Lagrangian may be expressed *in terms of the velocity potential* as follows:

$$\mathcal{L} = \frac{\rho_0}{4} \Box^2 \phi^2 \quad . \tag{3.15}$$

With the above relations, we may now readily express the second-order wave equation, Eq. (3.7), in terms of ϕ. Substitution into first and second-order terms involving the pressure using Eqs. (3.11) and (3.12), respectively, yields

$$\frac{\partial}{\partial t} \Box_0^2 \phi + \frac{b}{\rho_0 c_0^4} \frac{\partial^4 \phi}{\partial t^4} + \frac{m_0}{c_0^2} \int_{-\infty}^{t} \frac{\partial^4 \phi}{\partial y^4} e^{-(t-y)/\tau_0} \, dy$$
$$= \frac{2}{\rho_0 c_0^2} \frac{\partial^2}{\partial t^2} \mathcal{L} + \frac{\beta}{c_0^4} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \phi}{\partial t} \right)^2 \quad . \tag{3.16}$$

A higher-order term, $(\nu V / c_0^2) \Box^2 \phi$, was neglected in obtaining the above relation. In order to integrate the above equation, we rearrange the relaxation term using the following:

$$\int_{-\infty}^{t} \frac{\partial^4 \phi}{\partial y^4} e^{-(t-y)/\tau_0} \, dy = \frac{\partial}{\partial t} \int_{-\infty}^{t} \frac{\partial^3 \phi}{\partial y^3} e^{-(t-y)/\tau_0} \, dy \quad . \tag{3.17}$$

Equation (3.17) can be verified by integrating the left-hand side by parts and by applying Leibnitz's theorem to the right-hand side. Substituting Eq. (3.17) into Eq. (3.16) and then integrating with respect to time leads to

$$\Box_0^2\phi + \frac{b}{\rho_0 c_0^4}\frac{\partial^3\phi}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t}\frac{\partial^3\phi}{\partial y^3}e^{-(t-y)/\tau_0}\,dy = \frac{2}{\rho_0 c_0^2}\mathcal{L} + \frac{\beta}{c_0^4}\left(\frac{\partial\phi}{\partial t}\right)^2 \quad . \tag{3.18}$$

By expanding the nonlinearity terms using the definition of β and $\mathcal{L}$, Eqs. (3.4) and (3.13), we see that Eq. (3.18) may be written as

$$\Box_0^2\phi + \frac{b}{\rho_0 c_0^4}\frac{\partial^3\phi}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t}\frac{\partial^3\phi}{\partial y^3}e^{-(t-y)/\tau_0}\,dy = \frac{1}{c_0^2}\left[(\nabla\phi)^2 + \frac{1}{c_0^2}\frac{B}{2A}\left(\frac{\partial\phi}{\partial t}\right)^2\right] \quad . \tag{3.19}$$

Equations (3.18) and (3.19) are equivalent forms of the wave equation in terms of the velocity potential ϕ. The terms in Eq. (3.18) have the same meaning as the corresponding terms in Eq. (3.7).

3–4 Wave Equations in Terms of the Modified Pressure P' and the Modified Velocity Potential Φ

In this section, the wave equation is expressed in terms of two new variables, the modified acoustic pressure P' and the modified velocity potential Φ, which are defined below.

To motivate the definition of P' and to find the wave equation in terms of it, we start with the wave equation in terms of the acoustic pressure, Eq. (3.7). Equation (3.7) may be rearranged with the aid of Eq. (3.15) to obtain

$$\Box_0^2\left[p' + \frac{\rho_0}{4}\left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\phi^2\right] + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 p'}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t}\frac{\partial^3 p'}{\partial y^3}e^{-(t-y)/\tau_0}\,dy$$
$$= -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(p')^2 \quad . \tag{3.20}$$

The modified acoustic pressure P' is now defined as

$$P' \equiv p' + \frac{\rho_0}{4}\left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\phi^2 \quad . \tag{3.21}$$

To first order, the modified and the ordinary acoustic pressures are equal. The difference, a second-order term, accounts for local nonlinear effects. The wave

equation in terms of modified acoustic pressure is, accordingly,

$$\Box_0^2 P' + \frac{b}{\rho_0 c_0^4}\frac{\partial^3 P'}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t}\frac{\partial^3 P'}{\partial y^3}e^{-(t-y)/\tau_0}\,dy = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}(P')^2 \quad . \tag{3.22}$$

In Eq. (3.22), no single term explicitly accounts for the local effects. The effects are instead accounted for in the definition of the modified acoustic pressure.

An expression for the modified pressure P' in terms of the velocity potential ϕ motivates the definition of the modified velocity potential Φ. If Eq. (3.11) is substituted into the definition of P', Eq. (3.21), the following expression is obtained:

$$P' = -\rho_0\frac{\partial}{\partial t}\left(\phi - \frac{1}{2c_0^2}\frac{\partial}{\partial t}\phi^2\right) + \rho_0\nu V\nabla^2\phi \quad .$$

The modified velocity potential is now defined as

$$\Phi \equiv \phi - \frac{1}{2c_0^2}\frac{\partial}{\partial t}\phi^2 \quad . \tag{3.23}$$

Thus the modified acoustic pressure is related to the modified velocity potential by

$$P' = -\rho_0\frac{\partial\Phi}{\partial t} + \rho_0\nu V\nabla^2\Phi \quad . \tag{3.24}$$

Inserting Eq. (3.24) into Eq. (3.22) and integrating noting Eq. (3.17) leads to the wave equation in terms of the modified velocity potential,

$$\Box_0^2\Phi + \frac{b}{\rho_0 c_0^4}\frac{\partial^3\Phi}{\partial t^3} + \frac{m_0}{c_0^2}\int_{-\infty}^{t}\frac{\partial^3\Phi}{\partial t^3}e^{-(t-y)/\tau_0}\,dy = \frac{\beta}{c_0^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)^2 \quad . \tag{3.25}$$

As with the wave equation in terms of the modified pressure, Eq. (3.22), no single term in Eq. (3.25) accounts for local nonlinear effects. These effects are instead accounted for in the modified velocity potential Φ.

Equation (3.25) may be obtained directly from the wave equation in terms of the velocity potential, Eq. (3.19). Such an approach yields the same motivation for the definition of the modified velocity potential.

3–5 Summary

In this chapter, wave equations in terms of the acoustic pressure p', the velocity potential ϕ, and two new variables—the modified acoustic pressure

P' and the modified velocity potential Φ—were derived. They are, respectively, Eqs. (3.7), (3.19), (3.22), and (3.25). The wave equations in terms of the modified variables contain one less term than their ordinary counterparts because the modified variables themselves account for the local nonlinear effects.

CHAPTER 4

ON THE BOUNDARY CONDITIONS AT THE INTERFACE BETWEEN TWO LOSSLESS, IMMISCIBLE FLUIDS

4–1 Introduction

In this chapter, the boundary conditions at the interface between two inviscid, immiscible fluids are examined. The fluids are assumed to be initially quiet, and the interface is assumed to be initially planar and coincident with the $z = 0$ plane. Moreover, we neglect the effects of surface tension and body forces at the interface. Expressions that are correct to second order are obtained for each of the two boundary conditions—the kinematic condition and the force balance condition (Newton's second law). Although our analysis in later chapters is restricted to two dimensions (plane waves obliquely incident on an initially plane interface), the approximate forms of the boundary conditions developed in this chapter are for three dimensions.

The superscripts I and II are used to indicate the different fluids throughout the remainder of this dissertation.

4–2 Kinematic Condition

The author can think of no better way to express the kinematic boundary condition than to quote from Prandtl and Tietjens (1934),

> The kinematic boundary conditions at the surface of contact between a liquid and a solid body, and also between two immiscible fluids (water and oil, water and air, etc.), must clearly be such that neither vacuum nor interpenetration can occur. The necessary consequence of this is that the normal components of the velocities of the two media are equal on each side of the surface of contact, ...

Obtaining the normal component of the particle velocities at the interface requires knowledge of the normal to the interface. Consider Figure 4.1, which depicts the interface between fluids I and II at time t, where $t > 0$. Because the interface initially lies in the $z = 0$ plane, the equation for the interface may be written as

$$z - f(x, y, t) = 0 \quad , \tag{4.1}$$

or, alternatively, as

$$F(x, y, z, t) = 0 \quad . \tag{4.2}$$

A vector normal to the interface $\mathbf{N}(x, y, z, t)$ is [see, for example, Thomas and

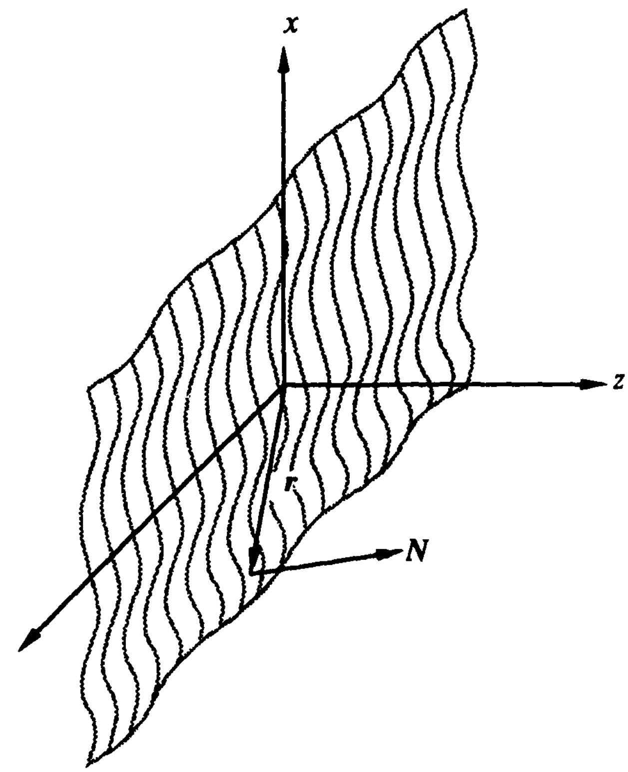

Figure 4.1 INTERFACE BETWEEN FLUIDS I AND II AT TIME $t > 0$

Finney (1979)]

$$\mathbf{N}(x, y, z, t) = -\mathbf{i}\,\frac{\partial f}{\partial x} - \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k} \quad , \tag{4.3}$$

or, in terms of the function F,

$$\mathbf{N} = \nabla F \quad . \tag{4.4}$$

Note that for later convenience $\mathbf{N}$ is used rather than the unit normal $\nabla F/|\nabla F|$. If the interface has a velocity of $\mathbf{u}_F$, then the normal velocity of the interface and the normal velocity of a particle at the interface on, say, the fluid I side must match,

$$\mathbf{u}_F \cdot \mathbf{N} = \mathbf{u}^{\mathrm{I}} \cdot \mathbf{N} \quad , \tag{4.5}$$

where $\mathbf{u}^{\mathrm{I}}$ is the velocity of fluid I. The situation is the same on the fluid II side,

$$\mathbf{u}_F \cdot \mathbf{N} = \mathbf{u}^{\mathrm{II}} \cdot \mathbf{N} \quad , \tag{4.6}$$

where $\mathbf{u}^{\mathrm{II}}$ is the velocity of fluid II. The kinematic boundary condition between the two inviscid, immiscible fluids may thus be written as follows:[1]

$$\mathbf{u}^{\mathrm{I}} \cdot \mathbf{N} = \mathbf{u}^{\mathrm{II}} \cdot \mathbf{N} \quad \text{on} \quad F(x,y,z,t) = 0 \quad . \tag{4.7}$$

For future reference, we note that $\mathbf{u}^{\mathrm{I}}$ and $\mathbf{u}^{\mathrm{II}}$ may be written in terms of their x, y, and z components as follows:

$$\mathbf{u}^{\mathrm{I}} = \mathbf{i}u^{\mathrm{I}} + \mathbf{j}v^{\mathrm{I}}, + \mathbf{k}w^{\mathrm{I}} \quad , \tag{4.8}$$

$$\mathbf{u}^{\mathrm{II}} = \mathbf{i}u^{\mathrm{II}} + \mathbf{j}v^{\mathrm{II}} + \mathbf{k}w^{\mathrm{II}} \quad . \tag{4.9}$$

It turns out that it is important to know what happens to a particle that is initially on the interface. Physically, we may reason that if the normal velocity of a fluid particle at the interface is equal to the normal velocity of the interface itself, then a particle that is initially on the interface must remain on the interface. This may be shown mathematically by following the developments of Myers (1980) and Lamb (1932). The velocity of a point on the interface, $\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$, is given by

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla F = 0 \quad \text{on} \quad F(x,y,z,t) = 0 \quad .$$

Noting that $d\mathbf{r}/dt = \mathbf{u}_F$ and substituting using Eq. (4.5) yields

$$\frac{\partial F}{\partial t} + \mathbf{u}^{\mathrm{I}} \cdot \nabla F = \frac{DF}{Dt} = 0 \quad \text{on} \quad F(x,y,z,t) = 0 \quad .$$

Following the proof given by Lord Kelvin, Lamb (1932, p. 7) shows that, for an inviscid fluid, an interface whose equation satisfies $DF/Dt = 0$ always consists of

[1]Note that if the fluid were viscous, a second kinematic condition would emerge. This kinematic condition would govern the particle velocity tangential to the interface.

the same fluid particles, that is, that a particle that is initially on the interface remains on the interface.

Note that the kinematic boundary condition as expressed in Eq. (4.7) is to be applied on the interface, the location of which is time varying. Consequently, the normal to the interface is also time varying. However, because the fluid is assumed initially quiet, any motion of the interface must be caused by the impinging acoustic signal. It would appear, therefore, to be possible to describe the motion of the normal to the interface in terms of the impinging acoustic signal. Since the kinematic boundary condition already depends directly on the particle velocity, an expression for the normal in terms of the particle velocity is sought.

It turns out, however, to be convenient to find first an expression for the normal in terms of the particle displacement. To that end, the expression for the normal to interface given in Eq. (4.3) is used, and expressions for $\partial f/\partial x$ and $\partial f/\partial y$ are obtained in terms of particle displacement. The displacement of a fluid particle that is at location x, y, z at time t is denoted $\boldsymbol{\xi}$,

$$\boldsymbol{\xi}(x,y,z,t) \equiv \mathbf{i}\,A(x,y,z,t) + \mathbf{j}\,B(x,y,z,t) + \mathbf{k}\,C(x,y,z,t) \quad , \tag{4.10}$$

where A, B, and C represent, respectively, the components of displacement from the initial (rest) position of the particle. The particle displacement and the particle velocity are related in the following manner:

$$\mathbf{u} = \frac{D\boldsymbol{\xi}}{Dt} \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\xi} \quad . \tag{4.11}$$

Since a particle that is initially on the interface stays on the interface, the displacement of the interface given by Eq. (4.1) and the z component of Eq. (4.10) for a particle on the interface must be equal, that is,

$$z = f(x,y,t) = C\,(x,y,z,t)|_{z=f(x,y,t)} \quad . \tag{4.12}$$

The following relationships between the derivatives of f and C may be readily derived:

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial C}{\partial x}}{1 - \frac{\partial C}{\partial z}} \quad , \tag{4.13}$$

$$\frac{\partial f}{\partial y} = \frac{\frac{\partial C}{\partial y}}{1 - \frac{\partial C}{\partial z}} \quad . \tag{4.14}$$

Inserting Eqs. (4.13) and (4.14) into Eq. (4.3) and rearranging yields the desired form of the normal in terms of the particle displacement,

$$\mathbf{N} = \left(1 - \frac{\partial C}{\partial z}\right)^{-1} \left[-\mathbf{i}\,\frac{\partial C}{\partial x} - \mathbf{j}\,\frac{\partial C}{\partial y} + \mathbf{k}\left(1 - \frac{\partial C}{\partial z}\right)\right]\Bigg|_{z=f(x,y,t)} \quad . \tag{4.15}$$

Since no approximations have been made, Eq. (4.15) is an exact expression for the normal to the moving interface.

Next, an expression for the normal that is correct to first order (linear in the acoustic variables) is obtained. Performing a binomial expansion of the denominator of Eq. (4.15) and retaining only linear terms leads to

$$\mathbf{N} = -\mathbf{i}\,\frac{\partial C}{\partial x} - \mathbf{j}\,\frac{\partial C}{\partial y} + \mathbf{k}\Bigg|_{z=f(x,y,t)} \quad . \tag{4.16}$$

Noting that the first-order approximation of Eq. (4.11) is

$$\mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{4.17}$$

and that the fluid is assumed to be initially at rest, we see that the first-order approximation of $C(x, y, z, t)$ is

$$C(x, y, z, t) = \int w(x, y, z, t)\, dt\Bigg|_{z=f(x,y,t)} \quad . \tag{4.18}$$

Use of Eq. (4.18) leads to the first-order approximation of the normal expressed in terms of the particle velocity,

$$\mathbf{N} = -\mathbf{i}\int \frac{\partial w}{\partial x}\, dt - \mathbf{j}\int \frac{\partial w}{\partial y}\, dt + \mathbf{k}\Bigg|_{z=f(x,y,t)} \quad . \tag{4.19}$$

An expression of the kinematic boundary condition applied *on the moving interface* that is correct to second order is now obtained. Substitution of Eq. (4.19) into the kinematic boundary condition, Eq. (4.7), yields the following expression, which is correct to second order:

$$\begin{aligned} -u^{\mathrm{I}} \int \frac{\partial w^{\mathrm{I}}}{\partial x}\, dt \; - \; v^{\mathrm{I}} \int \frac{\partial w^{\mathrm{I}}}{\partial y}\, dt + w^{\mathrm{I}} &= \\ - \; u^{\mathrm{II}} \int \frac{\partial w^{\mathrm{II}}}{\partial x}\, dt - v^{\mathrm{II}} \int \frac{\partial w^{\mathrm{II}}}{\partial y}\, dt + w^{\mathrm{II}} \quad \text{on } z = f(x, y, t) \quad . \end{aligned} \tag{4.20}$$

Since the first-order approximation of Eq. (4.20) is

$$w^{\mathrm{I}} = w^{\mathrm{II}} \quad \text{on } z = f(x,y,t) \quad ,$$

we see that Eq. (4.20) may be written

$$(u^{\mathrm{II}} - u^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial x}\, dt + (v^{\mathrm{II}} - v^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial y}\, dt + w^{\mathrm{I}} = w^{\mathrm{II}} \quad \text{on } z = f(x,y,t) \quad . \tag{4.21}$$

Equation (4.21) is the kinematic boundary condition applied *on the moving interface* correct to second order.

An expression, which is correct to second order, for the kinematic boundary condition to be applied at $z = 0$ rather than on the moving interface $z = f(x,y,t)$ is now obtained. This step is required because the wave equation is most conveniently solved subject to boundary conditions on boundaries that do not move. Application of a Taylor series expansion to Eq. (4.21) and retaining terms up to second order yields

$$\begin{aligned}(u^{\mathrm{II}} - u^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial x}\, dt \; + \; (v^{\mathrm{II}} - v^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial y}\, dt & \\ + \; w^{\mathrm{I}} + f(x,y,t)\, \frac{\partial w^{\mathrm{I}}}{\partial z} = w^{\mathrm{II}} + f(x,y,t)\, \frac{\partial w^{\mathrm{II}}}{\partial z} & \quad \text{on } z = 0 \quad .\end{aligned}$$

Substitution of the leading-order term in the Taylor series expansion of Eq. (4.12) into Eq. (4.18) leads to

$$f(x,y,t) = \int w(x,y,z,t)\, dt \bigg|_{z=0} \quad . \tag{4.22}$$

Combining the two foregoing relations yields

$$\begin{aligned}(u^{\mathrm{II}} - u^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial x}\, dt + (v^{\mathrm{II}} - v^{\mathrm{I}}) \int \frac{\partial w^{\mathrm{II}}}{\partial y}\, dt \; + \; \left(\frac{\partial w^{\mathrm{I}}}{\partial z} - \frac{\partial w^{\mathrm{II}}}{\partial z}\right) \int w^{\mathrm{II}}\, dt & \\ = w^{\mathrm{II}} - w^{\mathrm{I}} \quad \text{on } z = 0 \quad . & \end{aligned} \tag{4.23}$$

Equation (4.23) is the kinematic boundary condition that applies at $z = 0$ correct to second order. Dropping the second-order terms yields

$$w^{\mathrm{I}} = w^{\mathrm{II}} \quad \text{on } z = 0 \quad , \tag{4.24}$$

which is the well known small-signal form of the kinematic boundary condition: the normal particle velocities balance at the interface. Note that in the small-signal approximation the motion of the interface is assumed negligible.

4–3 Force Balance Condition

The second boundary condition—the force balance condition—is obtained by applying Newton's second law to the interface. Consider Fig. 4.2, which shows a material region (abbreviated MR) that spans the interface. Newton's second law may be expressed in a general form as follows [see, for example, Panton (1984, p. 107)]:

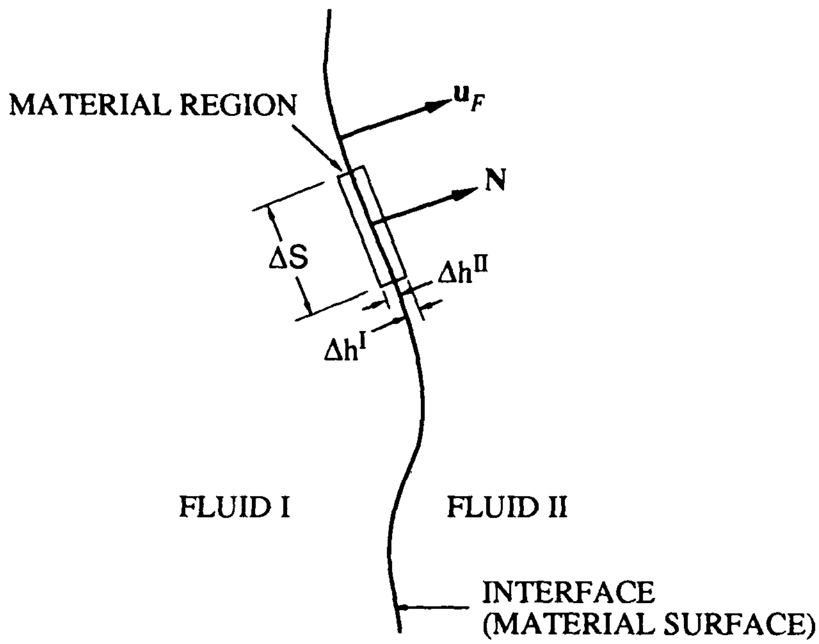

Figure 4.2 MATERIAL REGION THAT SPANS THE INTERFACE

$$\frac{d}{dt}\int_{\mathrm{MR}} \rho \mathbf{u}\, dV = \int_{\mathrm{MR}} \rho \mathbf{F}\, dV + \int_{\mathrm{MR}} \mathbf{R}\, dS \quad \text{on } z = f(x,y,t) \quad , \tag{4.25}$$

where $\mathbf{F}$ is the body force per unit mass vector, and $\mathbf{R}$ is the surface force per unit area vector. Since body forces (gravity, for example) are neglected, the first term on the right-hand side drops out. Moreover, the surface force vector reduces to the pressure applied normal to the material region because viscosity and surface tension are assumed negligible. If the material region is collapsed about the massless interface, the left-hand side of the Eq. (4.25) goes to zero, which leaves

$$p^{\mathrm{I}} = p^{\mathrm{II}} \quad \text{on } z = f(x,y,t) \quad . \tag{4.26}$$

Equation (4.26) is the exact boundary condition that is to be applied *on the moving interface.*

An expression for the pressure balance boundary condition on $z = 0$ that is correct to second order may be obtained by applying a Taylor series expansion to Eq. (4.26). Application of a Taylor series expansion and retention of terms up to second order yields

$$p^{\mathrm{I}} + f(x,y,t)\,\frac{\partial p^{\mathrm{I}}}{\partial z} = p^{\mathrm{II}} + f(x,y,t)\,\frac{\partial p^{\mathrm{II}}}{\partial z} \quad \text{on } z = 0 \quad .$$

Use of Eq. (4.22) in the above relation leads to the desired form of the pressure balance boundary condition on $z = 0$ that is correct to second order,

$$p^{\mathrm{II}} - p^{\mathrm{I}} = \left(\frac{\partial p^{\mathrm{I}}}{\partial z} - \frac{\partial p^{\mathrm{II}}}{\partial z}\right) \int w^{\mathrm{II}}\, dt \quad \text{on } z = 0 \quad . \tag{4.27}$$

In obtaining Eq. (4.27), the first-order form of the particle velocity balance has been used in the second-order term. Dropping the second-order terms in Eq. (4.27) leads to

$$p^{\mathrm{I}} = p^{\mathrm{II}} \quad \text{on } z = 0 \quad . \tag{4.28}$$

This is the well-known small-signal form of the pressure boundary condition: the pressures balance at the interface between two fluids.

CHAPTER 5

AN ANALYSIS OF THE REFLECTION AND REFRACTION OF FINITE-AMPLITUDE PLANE WAVES AT A PLANE FLUID-FLUID INTERFACE USING SECOND-ORDER PERTURBATION THEORY

5–1 Introduction

In this chapter we analyze the reflection and refraction of finite-amplitude plane waves that are obliquely incident on an initially plane fluid-fluid interface. A second-order perturbation analysis method, sometimes referred to as quasilinear analysis, is used. The notation and method closely follow the work of Naze Tjøtta and Tjøtta (1987). The boundary condition at the source is arbitrary. Special attention is given to the $O(\epsilon^2)$ source boundary condition and the motion of the interface. The chapter is divided as follows: In the first section, each acoustic variable is expanded in a power series in ϵ, where ϵ is a small parameter (the peak particle velocity normalized by c_0). The $O(\epsilon)$ and $O(\epsilon^2)$ interface boundary conditions, wave equations for fluids I and II, and relations between the various acoustic field variables are obtained. In the second section, the $O(\epsilon)$ system is solved, and expressions for the $O(\epsilon)$ reflected and transmitted acoustic fields are found. In the third section, we solve the $O(\epsilon^2)$ system and find expressions for the $O(\epsilon^2)$ reflected and transmitted acoustic fields. Although some interpretation of the results is given as they are obtained, the majority of interpretation of results is deferred to the next chapter.

It is useful to note at this time that oblique incidence reflection and refraction of plane waves is inherently a two-dimensional problem. Without loss of generality, we therefore take the incident ray to lie in the x, z plane. The subsequent motion—reflected wave, transmitted wave, and motion of the interface—is independent of the coordinate y. Thus, it is appropriate for our work to define

the propagation direction of the incoming signal as

$$\mathbf{n}^{\mathrm{inc}} = \mathbf{i}\sin\theta^{\mathrm{inc}} + \mathbf{k}\cos\theta^{\mathrm{inc}} \quad , \tag{5.1}$$

where, as above, θ is measured from the $+z$-axis; see Fig. 5.1. (Recall that the interface is assumed to initially lie in the $z = 0$ plane.) This orientation of the incident signal to the coordinate system is assumed through the remainder of this work. Moreover, all angles are measured from the $+z$-axis. The superscript inc is used throughout this work to indicate an acoustic variable associated with the incident signal; similarly, the superscripts refl and trans are used to indicate the reflected and transmitted signals.

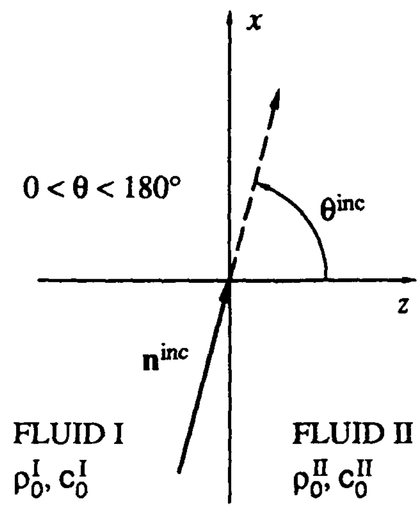

Figure 5.1 DIAGRAM SHOWING COORDINATES THAT ARE REQUIRED FOR OUR ANALYSIS

5–2 Basic Equations, Relations, and Boundary Conditions

Obtained in this section are $O(\epsilon)$ and $O(\epsilon^2)$ forms of the following: (1) the wave equations for fluids I and II, (2) the relations between the modified acoustic potential Φ and some of the other field variables, namely P', ϕ, p', and $\mathbf{u}$, and (3) the interface and source boundary conditions.

Wave equations of $O(\epsilon)$ and $O(\epsilon^2)$

The lossless version of the wave equation, Eq. (3.25), is

$$\Box^2\Phi = \frac{\beta}{c_0^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)^2 \quad . \tag{5.2}$$

To obtain $O(\epsilon)$ and $O(\epsilon^2)$ forms of Eq. (5.2), we expand the modified velocity potential Φ in a power series in terms of a small parameter ϵ (which is defined later from the source boundary condition),

$$\Phi = \epsilon\Phi_{(1)} + \epsilon^2\Phi_{(2)} + \cdots \quad . \tag{5.3}$$

If the expansion is inserted into Eq. (5.2) and if coefficients of like powers of ϵ are equated, then the following $O(\epsilon)$ and $O(\epsilon^2)$ forms of the wave equation for fluid I are obtained:

$$\Box^2\Phi^{\mathrm{I}}_{(1)} = 0 \quad , \tag{5.4}$$

$$\Box^2\Phi^{\mathrm{I}}_{(2)} = \frac{\beta^{\mathrm{I}}}{(c_0^{\mathrm{I}})^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi^{\mathrm{I}}_{(1)}}{\partial t}\right)^2 \quad . \tag{5.5}$$

Recall that the superscript I is used to indicate an acoustic variable or a fluid constant that is associated with fluid I. The corresponding pair of equations that govern the $O(\epsilon)$ and $O(\epsilon^2)$ acoustic fields in fluid II are

$$\Box^2\Phi^{\mathrm{II}}_{(1)} = 0 \quad , \tag{5.6}$$

$$\Box^2\Phi^{\mathrm{II}}_{(2)} = \frac{\beta^{\mathrm{II}}}{(c_0^{\mathrm{II}})^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi^{\mathrm{II}}_{(1)}}{\partial t}\right)^2 \quad . \tag{5.7}$$

$O(\epsilon)$ and $O(\epsilon^2)$ relations between Φ and P', ϕ, p', and u

Since the wave equation is expressed in terms of the modified acoustic potential Φ, the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the relations between Φ and some of the other field variables, namely, P', ϕ, p', and $\mathbf{u}$, are required. We start by expanding the modified acoustic pressure, the velocity potential, and the acoustic pressure in power series in ϵ:

$$P' = \epsilon P_{(1)} + \epsilon^2 P_{(2)} + \cdots \quad , \tag{5.8}$$

$$\phi = \epsilon\phi_{(1)} + \epsilon^2\phi_{(2)} + \cdots \quad , \tag{5.9}$$

$$p' = \epsilon p_{(1)} + \epsilon^2 p_{(2)} + \cdots \quad . \tag{5.10}$$

Note that for convenience the primes in the expanded forms of P' and p' have been dropped. The particle velocity $\mathbf{u}$ may be expanded in a power series as follows:

$$\mathbf{u} = \epsilon\mathbf{u}_{(1)} + \epsilon^2\mathbf{u}_{(2)} + \cdots \quad . \tag{5.11}$$

The $O(\epsilon)$ and $O(\epsilon^2)$ forms of $\mathbf{u}$ may be written in terms of their x and z components,

$$\mathbf{u}_{(1)} = \mathbf{i}u_{(1)} + \mathbf{k}w_{(1)} \quad , \tag{5.12}$$
$$\mathbf{u}_{(2)} = \mathbf{i}u_{(2)} + \mathbf{k}w_{(2)} \quad . \tag{5.13}$$

Alternatively, the components of $\mathbf{u}$ may be directly expanded as follows:

$$u = \epsilon u_{(1)} + \epsilon^2 u_{(2)} + \cdots \quad , \tag{5.14}$$
$$w = \epsilon w_{(1)} + \epsilon^2 w_{(2)} + \cdots \quad . \tag{5.15}$$

For the case of propagation in a lossless fluid, the expression for the modified acoustic pressure P' given in Eq. (3.24) simplifies to

$$P' = -\rho_0 \frac{\partial \Phi}{\partial t} \quad . \tag{5.16}$$

Inserting the expansions for P' and Φ into Eq. (5.16) and then equating coefficients of like powers of ϵ yields the following $O(\epsilon)$ and $O(\epsilon^2)$ relations for P' in fluids I and II:

$$P^{\mathrm{I}}_{(1)} = -\rho^{\mathrm{I}}_0 \frac{\partial \Phi^{\mathrm{I}}_{(1)}}{\partial t} \quad , \tag{5.17}$$
$$P^{\mathrm{I}}_{(2)} = -\rho^{\mathrm{I}}_0 \frac{\partial \Phi^{\mathrm{I}}_{(2)}}{\partial t} \quad , \tag{5.18}$$
$$P^{\mathrm{II}}_{(1)} = -\rho^{\mathrm{II}}_0 \frac{\partial \Phi^{\mathrm{II}}_{(1)}}{\partial t} \quad , \tag{5.19}$$
$$P^{\mathrm{II}}_{(2)} = -\rho^{\mathrm{II}}_0 \frac{\partial \Phi^{\mathrm{I}}_{(2)}}{\partial t} \quad . \tag{5.20}$$

The modified acoustic potential Φ is defined in Eq. (3.23) in terms of the acoustic potential ϕ. Using the expansions for Φ and ϕ leads to the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the relation between Φ and ϕ for fluids I and II,

$$\Phi^{\mathrm{I}}_{(1)} = \phi^{\mathrm{I}}_{(1)} \quad , \tag{5.21}$$

$$\Phi^{\mathrm{I}}_{(2)} = \phi^{\mathrm{I}}_{(2)} - \frac{1}{2(c_0^{\mathrm{I}})^2}\frac{\partial}{\partial t}(\phi^{\mathrm{I}}_{(1)})^2 \quad , \tag{5.22}$$

$$\Phi^{\mathrm{II}}_{(1)} = \phi^{\mathrm{II}}_{(1)} \quad , \tag{5.23}$$

$$\Phi^{\mathrm{II}}_{(2)} = \phi^{\mathrm{II}}_{(2)} - \frac{1}{2(c_0^{\mathrm{II}})^2}\frac{\partial}{\partial t}(\phi^{\mathrm{II}}_{(1)})^2 \quad . \tag{5.24}$$

Similarly, the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the relation between the modified velocity potential Φ and the acoustic pressure p' for both fluids I and II may be found. Using the expansions for p', P', and ϕ in the defining relation for P', Eq. (3.21), and then rearranging and substituting using Eqs. (5.17)-(5.20), we obtain

$$p^{\mathrm{I}}_{(1)} = -\rho_0^{\mathrm{I}}\frac{\partial}{\partial t}\Phi^{\mathrm{I}}_{(1)} \quad , \tag{5.25}$$

$$p^{\mathrm{I}}_{(2)} = -\rho_0^{\mathrm{I}}\frac{\partial}{\partial t}\Phi^{\mathrm{I}}_{(2)} - \frac{\rho_0^{\mathrm{I}}}{4}\left(\nabla^2 + \frac{1}{(c_0^{\mathrm{I}})^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\mathrm{I}}_{(1)})^2 \quad , \tag{5.26}$$

$$p^{\mathrm{II}}_{(1)} = -\rho_0^{\mathrm{II}}\frac{\partial}{\partial t}\Phi^{\mathrm{II}}_{(1)} \quad , \tag{5.27}$$

$$p^{\mathrm{II}}_{(2)} = -\rho_0^{\mathrm{II}}\frac{\partial}{\partial t}\Phi^{\mathrm{II}}_{(2)} - \frac{\rho_0^{\mathrm{II}}}{4}\left(\nabla^2 + \frac{1}{(c_0^{\mathrm{II}})^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\mathrm{II}}_{(1)})^2 \quad . \tag{5.28}$$

Use of Eq. (5.11) in Eq. (3.9) and substitution of Eqs. (5.21)-(5.24) into the results yields the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the relation between the modified velocity potential Φ and the particle velocity $\mathbf{u}$ for both fluids I and II:

$$\mathbf{u}^{\mathrm{I}}_{(1)} = \nabla\Phi^{\mathrm{I}}_{(1)} \quad , \tag{5.29}$$

$$\mathbf{u}^{\mathrm{I}}_{(2)} = \nabla\Phi^{\mathrm{I}}_{(2)} + \frac{1}{2(c_0^{\mathrm{I}})^2}\frac{\partial}{\partial t}\nabla(\Phi^{\mathrm{I}}_{(1)})^2 \quad , \tag{5.30}$$

$$\mathbf{u}^{\mathrm{II}}_{(1)} = \nabla\Phi^{\mathrm{II}}_{(1)} \quad , \tag{5.31}$$

$$\mathbf{u}^{\mathrm{II}}_{(2)} = \nabla\Phi^{\mathrm{II}}_{(2)} + \frac{1}{2(c_0^{\mathrm{II}})^2}\frac{\partial}{\partial t}\nabla(\Phi^{\mathrm{II}}_{(1)})^2 \quad . \tag{5.32}$$

$O(\epsilon)$ and $O(\epsilon^2)$ forms of the interface boundary conditions

The $O(\epsilon)$ and $O(\epsilon^2)$ forms of the two dimensional versions of the interface boundary conditions, Eqs. (4.23) and (4.27), may be obtained by substituting in the expansions for p', u, and w. Equating coefficients of like powers of ϵ yields the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the expression for the normal particle velocity balance

at the interface,

$$w^{\rm I}_{(1)}\Big|_{z=0} = w^{\rm II}_{(1)}\Big|_{z=0} \quad , \tag{5.33}$$

$$w^{\rm II}_{(2)}\Big|_{z=0} - w^{\rm I}_{(2)}\Big|_{z=0} = (u^{\rm II}_{(1)} - u^{\rm I}_{(1)})\int \frac{\partial w^{\rm II}}{\partial x}dt\Big|_{z=0} + \left(\frac{\partial w^{\rm I}_{(1)}}{\partial z} - \frac{\partial w^{\rm II}_{(1)}}{\partial z}\right)\int w^{\rm II}_{(1)}dt\Big|_{z=0} \quad , \tag{5.34}$$

and the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the pressure balance,

$$p^{\rm I}_{(1)}\Big|_{z=0} = p^{\rm II}_{(1)}\Big|_{z=0} \quad , \tag{5.35}$$

$$p^{\rm II}_{(2)}\Big|_{z=0} - p^{\rm I}_{(2)}\Big|_{z=0} = \left(\frac{\partial p^{\rm I}_{(1)}}{\partial z} - \frac{\partial p^{\rm II}_{(1)}}{\partial z}\right)\int w^{\rm II}_{(1)}dt\Big|_{z=0} \quad . \tag{5.36}$$

$O(\epsilon)$ and $O(\epsilon^2)$ forms of the source boundary condition

The boundary condition at the source is arbitrary, and our results may therefore be used to analyze transients or harmonic motion. Although, strictly speaking, our source must be infinite in extent because we have assumed infinite plane waves, we assume that the reflected signal does not interact with the active source. This is reasonable, however, because practical experiments that approximate plane waves can be devised. Moreover, the theory developed here may be used as a single component of the spatial Fourier decomposition of a directive source, in which case the geometry may be selected such that the reflected signal does not return to the active source.

The boundary condition at the source may be written as follows:

$$\mathbf{u} = -\frac{\mathbf{n}}{c_0^{\rm I}}\frac{\partial}{\partial t}S(t-\tau_0) \quad \text{at} \quad \mathbf{r} = \mathbf{r}_0 \quad , \tag{5.37}$$

where $\mathbf{r} = \mathbf{r}_0$ defines a plane, S is any given function of t, and τ_0 is defined as $\mathbf{n}\cdot\mathbf{r}_0/c_0^{\rm I}$. The coefficient, the time derivative, and the phase shift are introduced for later convenience, specifically, to make it easier to match the solution in terms of the modified velocity potential. Using the expansion of the particle velocity given in Eq. (5.11) and expanding S in a power series as

$$S = \epsilon S_{(1)} + \epsilon^2 S_{(2)} + \cdots \tag{5.38}$$

leads to the $O(\epsilon)$ and $O(\epsilon^2)$ forms of the arbitrary particle velocity source condition,

$$\mathbf{u}_{(1)} = -\frac{\mathbf{n}}{c_0^{\rm I}}\frac{\partial}{\partial t}S_{(1)}(t-\tau_0) \quad \text{at} \quad \mathbf{r} = \mathbf{r}_0 \quad , \tag{5.39}$$

$$\mathbf{u}_{(2)} = -\frac{\mathbf{n}}{c_0^{\mathrm{I}}} \frac{\partial}{\partial t} S_{(2)}(t - \tau_0) \quad \text{at} \quad \mathbf{r} = \mathbf{r}_0 \quad . \tag{5.40}$$

5–3 Solution of the $O(\epsilon)$ System

In this section, the $O(\epsilon)$ system is solved. The $O(\epsilon)$ system consists of the $O(\epsilon)$ wave equations for fluids I and II, Eqs. (5.4) and (5.6), and the $O(\epsilon)$ boundary conditions. The $O(\epsilon)$ boundary conditions consists of the $O(\epsilon)$ interface boundary conditions, Eqs. (5.33) and (5.35), and the $O(\epsilon)$ boundary condition at the source, Eq. (5.39). To these we add the Sommerfeld radiation conditions for fluids I and II, that is, that the reflected and transmitted waves propagate away from the interface in fluids I and II, respectively. First, the general solutions of the wave equations in fluids I and II are found. Next, application of the interface boundary condition leads first to the law of specular reflection and Snell's law as conditions for the validity of the solutions and second to the reflection and transmission coefficients. Finally, the incident signal is matched to the source boundary condition.

To solve the $O(\epsilon)$ wave equations, we first note that because they are homogenous, no particular solutions are required. If the reflected and transmitted waves are assumed to be planar, the general solution of the $O(\epsilon)$ wave equations for fluids I and II may be written as follows:

$$\Phi^{\mathrm{I}}_{(1)}(\mathbf{r}, t) = \Phi^{\mathrm{inc}}_{(1)}(\tau^{\mathrm{inc}}) + \Phi^{\mathrm{refl}}_{(1)}(\tau^{\mathrm{refl}}_{(1)}) \quad , \tag{5.41}$$

$$\Phi^{\mathrm{II}}_{(1)}(\mathbf{r}, t) = \Phi^{\mathrm{trans}}_{(1)}(\tau^{\mathrm{trans}}_{(1)}) \quad . \tag{5.42}$$

The definitions of τ^{inc}, $\tau^{\mathrm{refl}}_{(1)}$, and $\tau^{\mathrm{trans}}_{(1)}$ are

$$\tau^{\mathrm{inc}} \equiv t - \frac{\mathbf{n}^{\mathrm{inc}} \cdot \mathbf{r}}{c_0^{\mathrm{I}}} \quad , \tag{5.43}$$

$$\tau^{\mathrm{refl}}_{(1)} \equiv t - \frac{\mathbf{n}^{\mathrm{refl}}_{(1)} \cdot \mathbf{r}}{c_0^{\mathrm{I}}} \quad , \tag{5.44}$$

$$\tau^{\mathrm{trans}}_{(1)} \equiv t - \frac{\mathbf{n}^{\mathrm{trans}}_{(1)} \cdot \mathbf{r}}{c_0^{\mathrm{II}}} \quad , \tag{5.45}$$

where $\mathbf{n}^{\mathrm{refl}}_{(1)}$ and $\mathbf{n}^{\mathrm{trans}}_{(1)}$ are unit vectors in the direction of propagation of the

reflected and transmitted waves, respectively (see Fig. 5.2),[1]

$$\mathbf{n}^{\mathrm{ref}}_{(1)} = \mathbf{i}\,\sin\theta^{\mathrm{refl}}_{(1)} + \mathbf{k}\,\cos\theta^{\mathrm{refl}}_{(1)} \quad \text{where} \quad \cos\theta^{\mathrm{refl}}_{(1)} \le 0 \quad , \tag{5.46}$$

$$\mathbf{n}^{\mathrm{trans}}_{(1)} = \mathbf{i}\,\sin\theta^{\mathrm{trans}}_{(1)} + \mathbf{k}\,\cos\theta^{\mathrm{trans}}_{(1)} \quad \text{where} \quad \cos\theta^{\mathrm{trans}}_{(1)} \ge 0 \quad . \tag{5.47}$$

The assumption that the reflected and transmitted waves are planar is tested

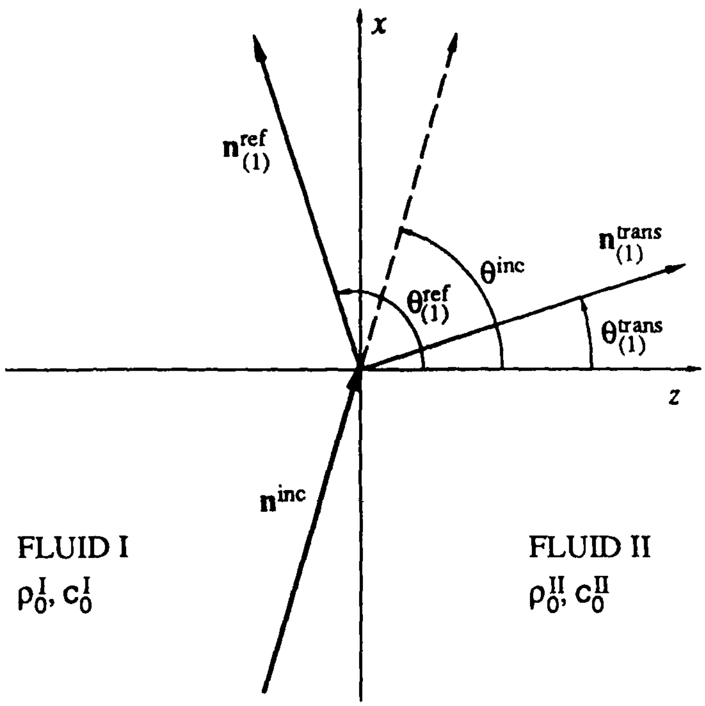

Figure 5.2 DIAGRAM SHOWING THE ANGLE OF INCIDENCE AND THE $O(\epsilon)$ ANGLES OF REFLECTION AND TRANSMISSION

by checking whether the assumed solution satisfies the interface boundary conditions.

Use of Eqs. (5.41) and (5.42) in the $O(\epsilon)$ boundary conditions at the interface, Eqs. (5.33) and (5.35), respectively, leads to

$$\frac{\cos\theta^{\mathrm{refl}}_{(1)}}{c_0^{\mathrm{I}}}\,\frac{\partial\Phi^{\mathrm{refl}}_{(1)}}{\partial t}\bigg|_{z=0} - \frac{\cos\theta^{\mathrm{trans}}_{(1)}}{c_0^{\mathrm{II}}}\,\frac{\partial\Phi^{\mathrm{trans}}_{(1)}}{\partial t}\bigg|_{z=0} = -\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\,\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\bigg|_{z=0} \tag{5.48}$$

[1] This nonstandard definition of the reflection angle θ^{refl} is chosen to simplify our later work. It turns out to be easier to manage the problem if all angles (more are introduced in the $O(\epsilon^2)$ analysis) are defined relative to the same reference; we chose the $+z$-axis.

and

$$\left.\frac{\partial \Phi_{(1)}^{\text{refl}}}{\partial t}\right|_{z=0} - \frac{\rho_0^{\text{II}}}{\rho_0^{\text{I}}} \left.\frac{\partial \Phi_{(1)}^{\text{trans}}}{\partial t}\right|_{z=0} = \left.\frac{\partial \Phi_{(1)}^{\text{inc}}}{\partial t}\right|_{z=0} \quad , \tag{5.49}$$

where we have used the following chain rule result:

$$\nabla \Phi_{(1)} = -\frac{\mathbf{n}}{c_0} \frac{\partial \Phi_{(1)}}{\partial t} \quad . \tag{5.50}$$

This relation is used repeatedly in later analysis. Equations (5.48) and (5.49) are written with the $O(\epsilon)$ incident signal on the right-hand side because it is a 'known' function. The form of the $O(\epsilon)$ incident signal is obtained by matching the $O(\epsilon)$ source condition, which is deferred until later.

Snell's law and the law of specular reflection are now obtained as conditions for the validity of the solutions. Our derivation of Snell's law and the law of specular reflection is a little longer than some other more common approaches. This approach is, however, the same one that we use to solve the $O(\epsilon^2)$ problem. It is therefore used here so that the reader may become familiar with it. Equations (5.48) and (5.49) represent a system of two equations and two unknowns with two conditions to be met, namely, that the solutions be valid for all x and all t. The conditions may be obtained by taking the x and t derivatives of the pair of equations, solving each set independently, then forcing the solutions to be the same. But if the solutions are to be the same, then the equations generated by taking the x and t derivatives of the boundary conditions must be linearly dependent. Accordingly, we arrive at the following conditions by forcing the x and t derivatives of Eqs. (5.48) and (5.49) to be linearly dependent:

$$\sin\theta^{\text{inc}} = \sin\theta_{(1)}^{\text{refl}} \quad , \tag{5.51}$$

$$\sin\theta^{\text{inc}} = n \sin\theta_{(1)}^{\text{trans}} \quad , \tag{5.52}$$

where n is the ratio of the sound speeds,

$$n \equiv \frac{c_0^{I}}{c_0^{\text{II}}} \quad . \tag{5.53}$$

Equation (5.52) is Snell's law, and, as we soon see, Eq. (5.51) leads to the law of specular reflection.

We now solve for the angles $\theta_{(1)}^{\text{refl}}$ and $\theta_{(1)}^{\text{trans}}$. To satisfy Eqs. (5.51) and (5.46) simultaneously, θ^{inc} and $\theta_{(1)}^{\text{refl}}$ must be related as follows:

$$\theta_{(1)}^{\text{refl}} = 180^\circ - \theta^{\text{inc}} \quad . \tag{5.54}$$

To satisfy Eqs. (5.52) and (5.47) simultaneously, $\theta_{(1)}^{\text{trans}}$ and θ^{inc} must be related by

$$\theta_{(1)}^{\text{trans}} = \sin^{-1}(n^{-1}\sin\theta^{\text{inc}}) \quad . \tag{5.55}$$

If $n \geq 1$, $\theta_{(1)}^{\text{trans}}$ is real for all θ^{inc}; if $n < 1$, $\theta_{(1)}^{\text{trans}}$ is real provided that $|\sin\theta^{\text{inc}}| < n$. The angle of incidence at which $|\sin\theta^{\text{inc}}| = n$ is referred to as the critical angle. In what follows we assume for simplicity that $\theta_{(1)}^{\text{trans}}$ is real. The case of $\theta_{(1)}^{\text{trans}}$ being complex (incidence above critical angle) is briefly discussed in Sec. 5-5.

Use of Eqs. (5.54) and (5.55) in Eqs. (5.43), (5.44), and (5.45) shows that the values of τ^{inc}, $\tau_{(1)}^{\text{refl}}$, and $\tau_{(1)}^{\text{trans}}$ are equal when evaluated at the interface. For convenience, a new term that is equal to τ^{inc}, $\tau_{(1)}^{\text{refl}}$, and $\tau_{(1)}^{\text{trans}}$ when they are evaluated at the interface is now introduced,

$$\tau_\theta \equiv t - \frac{x\sin\theta^{\text{inc}}}{c_0^{\text{I}}} \quad . \tag{5.56}$$

The reflection and transmission coefficients, denoted R and T, respectively, are now introduced, and the two interface boundary conditions are used to yield expressions for the coefficients. The expressions depend only on the physical properties of the fluid and the angle of incidence. The coefficients R and T relate the reflected and transmitted pressures[2] at the interface to the incident pressure at the interface and are defined as follows:

$$R \equiv \frac{p_{(1)}^{\text{refl}}\Big|_{z=0}}{p_{(1)}^{\text{inc}}\Big|_{z=0}} \quad , \tag{5.57}$$

$$T \equiv \frac{p_{(1)}^{\text{trans}}\Big|_{z=0}}{p_{(1)}^{\text{inc}}\Big|_{z=0}} \quad , \tag{5.58}$$

where $p_{(1)}^{\text{inc}}$, $p_{(1)}^{\text{refl}}$, and $p_{(1)}^{\text{trans}}$ are the pressures corresponding to $\Phi_{(1)}^{\text{inc}}$, $\Phi_{(1)}^{\text{refl}}$, and $\Phi_{(1)}^{\text{trans}}$, respectively. At this time, we also define the acoustic impedance of fluids I and II, respectively,

$$Z^{\text{I}} \equiv \frac{\rho_0^{\text{I}} c_0^{\text{I}}}{\cos\theta^{\text{inc}}} \quad , \tag{5.59}$$

$$Z^{\text{II}} \equiv \frac{\rho_0^{\text{II}} c_0^{\text{II}}}{\cos\theta_{(1)}^{\text{trans}}} \quad . \tag{5.60}$$

[2]Some authors use coefficients based on the acoustic power of the incident, reflected, and transmitted waves. For clarity, our coefficients are sometimes referred to as amplitude reflection and transmission coefficients.

Note that the ratio of impedances is, accordingly,

$$\frac{Z^{\mathrm{I}}}{Z^{\mathrm{II}}} = \frac{n}{m} \frac{\cos\theta_{(1)}^{\mathrm{trans}}}{\cos\theta^{\mathrm{inc}}} \quad , \tag{5.61}$$

where m is the ratio of the densities (not the relaxation variable, which is no longer required),

$$m \equiv \frac{\rho_0^{\mathrm{II}}}{\rho_0^{\mathrm{I}}} \quad . \tag{5.62}$$

Note that m is defined as (fluid II density/fluid I density) whereas n is reciprocally defined (fluid I sound speed/fluid II sound speed). The reason for the difference is merely convenience. Use of Eqs. (5.57) and (5.58) in Eqs. (5.48) and (5.49) yields

$$1 - R = \frac{n}{m} \frac{\cos\theta_{(1)}^{\mathrm{trans}}}{\cos\theta^{\mathrm{inc}}} T \tag{5.63}$$

and

$$1 + R = T \quad , \tag{5.64}$$

where we have noted that

$$\cos\theta_{(1)}^{\mathrm{refl}} = -\cos\theta^{\mathrm{inc}} \quad . \tag{5.65}$$

Manipulations of Eqs. (5.63) and (5.64) yield the traditional results

$$R = \frac{Z^{\mathrm{II}} - Z^{\mathrm{I}}}{Z^{\mathrm{II}} + Z^{\mathrm{I}}} \tag{5.66}$$

and

$$T = \frac{2Z^{\mathrm{II}}}{Z^{\mathrm{I}} + Z^{\mathrm{II}}} \quad , \tag{5.67}$$

or, in terms of n and m,

$$R = \frac{m\cos\theta^{\mathrm{inc}} - n\cos\theta_{(1)}^{\mathrm{trans}}}{m\cos\theta^{\mathrm{inc}} + n\cos\theta_{(1)}^{\mathrm{trans}}} \quad , \tag{5.68}$$

$$T = \frac{2m\cos\theta^{\mathrm{inc}}}{m\cos\theta^{\mathrm{inc}} + n\cos\theta_{(1)}^{\mathrm{trans}}} \quad . \tag{5.69}$$

We now point out some interesting special cases. Note that for intromission (no $O(\epsilon)$ reflection) to occur, the acoustic impedances of the two fluids must be the same, that is, $Z^{\mathrm{I}} = Z^{\mathrm{II}}$, or, by Eq. (5.61),

$$\text{intromission :} \qquad \cos\theta_{(1)}^{\mathrm{trans}} = \frac{m}{n}\cos\theta^{\mathrm{inc}} \quad . \tag{5.70}$$

By squaring both sides of Eq. (5.70), eliminating the cosines in favor of sines, and using Snell's law, we see that

$$\sin^2\theta^{\text{intro}} = \frac{m^2-n^2}{m^2-1} \quad , \tag{5.71}$$

where $\theta^{\text{intro}} \equiv \theta^{\text{inc}}\Big|_{\text{intromission}}$. Note that for a intromission angle to exist, the following condition must be met:

$$\frac{m^2-n^2}{m^2-1} > 0 \quad .$$

Another special case occurs when the impedance of fluid II is very small in comparison to that of fluid I. The reflection coefficient tends to -1, and the interface is referred to as a pressure release interface. If, on the other hand, the impedance of fluid II is very large in comparison to that of fluid I, the reflection coefficient tends to unity. In this case, the interface is referred to as rigid. Note that although the transmission coefficient tends to 2 in this case, it can be shown that the acoustic power transmitted tends to zero.

We now obtain the reflected and transmitted fields in terms of the incident field. Note that the definitions of R and T may be rearranged to yield

$$\frac{\partial}{\partial t}\Phi^{\text{refl}}_{(1)}(\tau_\theta) = R\frac{\partial}{\partial t}\Phi^{\text{inc}}_{(1)}(\tau_\theta)$$

and

$$\frac{\partial}{\partial t}\Phi^{\text{trans}}_{(1)}(\tau_\theta) = \frac{T}{m}\frac{\partial}{\partial t}\Phi^{\text{inc}}_{(1)}(\tau_\theta) \quad .$$

The solutions away from the interface are obtained by replacing the independent variable τ_θ with $\tau^{\text{refl}}_{(1)}$ for the reflected field and with $\tau^{\text{trans}}_{(1)}$ for the transmitted field. The resulting field relations may be integrated once with respect to time noting that the integration constant must be zero in order to satisfy the condition that the reflected and transmitted fields be zero when the incident field is zero. Thus, away from the interface, the reflected and transmitted fields are

$$\Phi^{\text{refl}}_{(1)}(\tau^{\text{refl}}_{(1)}) = R\,\Phi^{\text{inc}}_{(1)}(\tau^{\text{refl}}_{(1)}) \quad , \tag{5.72}$$

$$\Phi^{\text{trans}}_{(1)}(\tau^{\text{trans}}_{(1)}) = \frac{T}{m}\Phi^{\text{inc}}_{(1)}(\tau^{\text{trans}}_{(1)}) \quad . \tag{5.73}$$

We now obtain an expression for $\Phi^{\text{inc}}_{(1)}$ that matches the source boundary condition. Because the reflected wave is assumed not to return to the active

source, the reflected wave is not involved in matching the source boundary condition. We may therefore write

$$\nabla\Phi^{\mathrm{I}}_{(1)}\Big|_{\mathbf{r}=\mathbf{r}_0} = -\frac{\mathbf{n}^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\Big|_{\mathbf{r}=\mathbf{r}_0} = -\frac{\mathbf{n}}{c_0^{\mathrm{I}}}\frac{\partial S_{(1)}}{\partial t} \quad . \tag{5.74}$$

Accordingly, to match the source condition, the following must be true:

$$\mathbf{n}^{\mathrm{inc}} = \mathbf{n} \quad , \tag{5.75}$$

$$\frac{\partial}{\partial t}\Phi^{\mathrm{inc}}_{(1)}(\tau^{\mathrm{inc}})\Big|_{\mathbf{r}=\mathbf{r}_0} = \frac{\partial}{\partial t}S_{(1)}(t-\tau_0) \quad . \tag{5.76}$$

Since, when evaluated at $\mathbf{r} = \mathbf{r}_0$, the independent variables of both functions are equal, the solution away from the interface may be obtained by replacing the independent variable with τ^{inc},

$$\frac{\partial}{\partial t}\Phi^{\mathrm{inc}}_{(1)}(\tau^{\mathrm{inc}}) = \frac{\partial}{\partial t}S_{(1)}(\tau^{\mathrm{inc}}) \quad .$$

To obtain an expression for $\Phi^{\mathrm{inc}}_{(1)}$ rather than its time derivative, we must integrate the foregoing relation with respect to time. This would, in general, mean that an arbitrary function that is linearly dependent on $\mathbf{r}$, say $\mathcal{F}(\mathbf{r})$, must be determined. We know, however, that the boundary condition at the source requires that $\nabla\mathcal{F}|_{\mathbf{r}=\mathbf{r}_0}$ be zero. The function $\mathcal{F}(\mathbf{r})$ must, therefore, be a constant. Since $\Phi_{(1)}$ is not a measurable quantity and all measurable quantities depend on derivatives of $\Phi_{(1)}$, we are free to choose this constant such that $\Phi_{(1)}$ at the source at time $t = 0$ is zero. We may, therefore, write the incident field as

$$\Phi^{\mathrm{inc}}_{(1)}(\tau^{\mathrm{inc}}) = S_{(1)}(\tau^{\mathrm{inc}}) - S_{(1)}(\tau_0) \quad . \tag{5.77}$$

Accordingly, the reflected and transmitted fields are

$$\Phi^{\mathrm{refl}}_{(1)} = R\left(S_{(1)}(\tau^{\mathrm{refl}}_{(1)}) - S_{(1)}(\tau_0)\right) \quad , \tag{5.78}$$

$$\Phi^{\mathrm{trans}}_{(1)} = \frac{T}{m}\left(S_{(1)}(\tau^{\mathrm{trans}}_{(1)}) - S_{(1)}(\tau_0)\right) \quad . \tag{5.79}$$

To summarize, we have derived the following major results: (1) Snell's law and the law of specular reflection, Eqs. (5.52) and (5.54), respectively, (2) the expressions for the reflection and transmission coefficients, Eqs. (5.66) and (5.67), respectively, and (3) solutions representing the $O(\epsilon)$ incident, reflected, and transmitted fields, Eqs. (5.77), (5.78), and (5.79), respectively.

5-4 Solution of the $O(\epsilon^2)$ System

In this section the $O(\epsilon^2)$ system is solved. The $O(\epsilon^2)$ system consists of the $O(\epsilon^2)$ wave equations for fluids I and II, Eqs. (5.5) and (5.7), respectively, the $O(\epsilon^2)$ boundary conditions at the interface, Eqs. (5.34) and (5.36), respectively, the Sommerfeld radiation conditions for fluids I and II, and the $O(\epsilon^2)$ boundary condition at the source, Eq. (5.40). The procedure is as follows: First, the general solutions for fluids I and II are separated into two parts—the particular solution, denoted by the subscript p, and the homogeneous solution, denoted by the subscript h:

$$\Phi^{\mathrm{I}}_{(2)} = \Phi^{\mathrm{I}}_{(2)p} + \Phi^{\mathrm{I}}_{(2)h} \quad , \tag{5.80}$$

$$\Phi^{\mathrm{II}}_{(2)} = \Phi^{\mathrm{II}}_{(2)p} + \Phi^{\mathrm{II}}_{(2)h} \quad . \tag{5.81}$$

The particular solutions are chosen to satisfy the $O(\epsilon^2)$ wave equations. The general solutions are then substituted into the $O(\epsilon^2)$ interface boundary conditions, and the $O(\epsilon^2)$ homogeneous solutions are chosen to satisfy these boundary conditions. Terms accounting for the finite displacement of the interface and the variation of the normal to the interface are identified. The results are then summarized.

The $O(\epsilon^2)$ particular solution

Since the $O(\epsilon)$ solution is known, it may be substituted into the right-hand side of the $O(\epsilon^2)$ wave equations. However, since the final form of the $O(\epsilon)$ solution depends on the boundary condition at the source, the functions $\Phi^{\mathrm{inc}}_{(1)}$, $\Phi^{\mathrm{refl}}_{(1)}$, and $\Phi^{\mathrm{trans}}_{(1)}$ are used. The $O(\epsilon^2)$ wave equations, Eqs. (5.5) and (5.7), then become

$$\Box^2(\Phi^{\mathrm{I}}_{(2)p}+\Phi^{\mathrm{I}}_{(2)h}) = \frac{\beta^{\mathrm{I}}}{(c_0^{\mathrm{I}})^4}\frac{\partial}{\partial t}\left[\left(\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\right)^2 + 2\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\frac{\partial\Phi^{\mathrm{refl}}_{(1)}}{\partial t} + \left(\frac{\partial\Phi^{\mathrm{refl}}_{(1)}}{\partial t}\right)^2\right] \quad , \tag{5.82}$$

$$\Box^2(\Phi^{\mathrm{II}}_{(2)p} + \Phi^{\mathrm{II}}_{(2)h}) = \frac{\beta^{\mathrm{II}}}{(c_0^{\mathrm{II}})^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi^{\mathrm{trans}}_{(1)}}{\partial t}\right)^2 \quad . \tag{5.83}$$

These equations are significantly more complicated than their $O(\epsilon)$ counterparts. The nonlinear terms on the right-hand side of Eq. (5.82) represent, respectively, the self-action of the incident signal, the interaction between the incident and reflected signals, and the self-action of the reflected signal. The nonlinear term

on the right-hand side of Eq. (5.83) represents the self-action of the transmitted signal.

Because the wave equation for fluid II is simpler, we solve for its particular solution first. Note that the right-hand side of Eq. (5.83) may be expressed as a d'Alembertian,

$$\frac{\beta^{\mathrm{II}}}{(c_0^{\mathrm{II}})^4}\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}^{\mathrm{trans}}}{\partial t}\right)^2 = -\Box^2\left[\frac{\beta^{\mathrm{II}}}{2(c_0^{\mathrm{II}})^3}\left(\mathbf{n}_{(1)}^{\mathrm{trans}}\cdot\mathbf{r}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{trans}}}{\partial t}\right)^2\right] \quad . \tag{5.84}$$

Equation (5.83) may thus be written

$$\Box^2\left[\Phi_{(2)p}^{\mathrm{II}} + \Phi_{(2)h}^{\mathrm{II}} + \frac{\beta^{\mathrm{II}}}{2(c_0^{\mathrm{II}})^3}\left(\mathbf{n}_{(1)}^{\mathrm{trans}}\cdot\mathbf{r}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{trans}}}{\partial t}\right)^2\right] = 0 \quad . \tag{5.85}$$

The particular solution for fluid II is now obvious,

$$\Phi_{(2)p}^{\mathrm{II}} = -\frac{\beta^{\mathrm{II}}}{2(c_0^{\mathrm{II}})^3}\left(\mathbf{n}_{(1)}^{\mathrm{trans}}\cdot\mathbf{r}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{trans}}}{\partial t}\right)^2 \quad . \tag{5.86}$$

The remaining equation,

$$\Box^2\Phi_{(2)h}^{\mathrm{II}} = 0 \quad , \tag{5.87}$$

is to be solved subject to the Sommerfeld radiation condition for fluid II and the $O(\epsilon^2)$ interface boundary conditions. We defer this to later. Note that the particular solution for fluid II exhibits amplitude growth in the direction of propagation via the coefficient $(\mathbf{n}_{(1)}^{\mathrm{trans}}\cdot\mathbf{r})$. Moreover, note that the direction of propagation of the particular solution for fluid II is given by Snell's law.

To obtain the particular solution of the wave equation for fluid I, we follow the method of Naze Tjøtta and Tjøtta (1987). As noted earlier, the particular solution may be thought of as being composed of three parts: the self-action of the incident wave, the self-action of the reflected wave, and the interaction between the incident and reflected waves. The fluid I wave equation is first rearranged to find the particular solution that pertains to the interaction between the incident and reflected waves. Noting that $\Box^2(\Phi_{(1)}^{\mathrm{inc}})^2 = 0$ and that $\Box^2(\Phi_{(1)}^{\mathrm{refl}})^2 = 0$ leads to the following form of $\Box^2(\Phi_{(1)}^{\mathrm{I}})^2$:

$$\Box^2(\Phi_{(1)}^{\mathrm{I}})^2 = 2\Box^2(\Phi_{(1)}^{\mathrm{inc}}\Phi_{(1)}^{\mathrm{refl}}) = \frac{2}{(c_0^{\mathrm{I}})^2}(\mathbf{n}^{\mathrm{inc}}\cdot\mathbf{n}_{(1)}^{\mathrm{refl}} - 1)\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\frac{\partial\Phi_{(1)}^{\mathrm{refl}}}{\partial t} \quad . \tag{5.88}$$

Use of the definitions of $\mathbf{n}^{\mathrm{inc}}$ and $\mathbf{n}^{\mathrm{refl}}_{(1)}$, the law of specular reflection, and some algebra simplifies Eq. (5.88) to

$$\frac{\partial \Phi^{\mathrm{inc}}_{(1)}}{\partial t}\frac{\partial \Phi^{\mathrm{refl}}_{(1)}}{\partial t} = -\frac{(c_0^{\mathrm{I}})^2}{4\cos^2\theta^{\mathrm{inc}}}\Box^2(\Phi^{\mathrm{inc}}_{(1)}\Phi^{\mathrm{refl}}_{(1)}) \quad . \tag{5.89}$$

Note that Eq. (5.89) may be expressed in terms of the Lagrangian $\mathcal{L}$, which was defined in Eq. (2.83). The $O(\epsilon)$ values of the Lagrangian in both fluids I and II are zero. The $O(\epsilon^2)$ value of the Lagrangian in fluid II is also zero. However, the $O(\epsilon^2)$ value of the Lagrangian in fluid I is

$$\mathcal{L}^{I}_{(2)} = \frac{\rho_0^{\mathrm{I}}}{2}\Box^2\left(\Phi^{\mathrm{inc}}_{(1)}\Phi^{\mathrm{refl}}_{(1)}\right) \quad . \tag{5.90}$$

With Eq. (5.89), the particular solution that pertains to the interaction between the incident and reflected waves is at hand. The inhomogeneous terms in Eq. (5.82) that correspond to the self-action of the incident and reflected waves are of the same form as the inhomogeneous term in the fluid II wave equation. Thus, they too may be expressed as d'Alembertians that are similar in form to Eq. (5.84). Using this idea and Eq. (5.89) in Eq. (5.82) leads to the following:

$$\Box^2\left[\Phi^{\mathrm{I}}_{(2)h} + \Phi^{\mathrm{I}}_{(2)p} + \frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^3}(\mathbf{n}^{\mathrm{inc}}\cdot\mathbf{r})\left(\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\right)^2 \right. \\ \left. + \frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2\cos^2\theta^{\mathrm{inc}}}\frac{\partial}{\partial t}(\Phi^{\mathrm{inc}}_{(1)}\Phi^{\mathrm{refl}}_{(1)}) + \frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^3}(\mathbf{n}^{\mathrm{refl}}_{(1)}\cdot\mathbf{r})\left(\frac{\partial\Phi^{\mathrm{refl}}_{(1)}}{\partial t}\right)^2\right] = 0 \quad . \tag{5.91}$$

The $O(\epsilon^2)$ particular solution for fluid I is therefore

$$\Phi^{\mathrm{I}}_{(2)p} = -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^3}\left[(\mathbf{n}^{\mathrm{inc}}\cdot\mathbf{r})\left(\frac{\partial\Phi^{\mathrm{inc}}_{(1)}}{\partial t}\right)^2 \right. \\ \left. + \frac{c_0^{\mathrm{I}}}{\cos^2\theta^{\mathrm{inc}}}\frac{\partial}{\partial t}(\Phi^{\mathrm{inc}}_{(1)}\Phi^{\mathrm{refl}}_{(1)}) + (\mathbf{n}^{\mathrm{refl}}_{(1)}\cdot\mathbf{r})\left(\frac{\partial\Phi^{\mathrm{refl}}_{(1)}}{\partial t}\right)^2\right] . \tag{5.92}$$

The remaining equation,

$$\Box^2\Phi^{\mathrm{I}}_{(2)h} = 0 \quad , \tag{5.93}$$

must be solved subject to the $O(\epsilon^2)$ interface boundary conditions and the $O(\epsilon^2)$ source condition. Note that the two terms in the particular solution for fluid

I that correspond to the self-action of the incident and reflected waves exhibit amplitude growth in the direction of propagation. The amplitude growth is given by the coefficients $(\mathbf{n}^{\text{inc}} \cdot \mathbf{r})$ and $(\mathbf{n}^{\text{refl}}_{(1)} \cdot \mathbf{r})$. Moreover, note that the direction of propagation of the term that corresponds to the self-action of the reflected wave is given the law of specular reflection.

The $O(\epsilon^2)$ homogeneous solution

Now that the particular solutions to the $O(\epsilon^2)$ wave equations are known, we may find the $O(\epsilon^2)$ homogeneous solutions. We start by examining the $O(\epsilon^2)$ interface boundary conditions and introducing some new notation. The general solutions for fluids I and II are then substituted into the $O(\epsilon^2)$ interface boundary conditions, and two relations between the $O(\epsilon^2)$ homogenous solutions are obtained. Next, the relations are solved for the $O(\epsilon^2)$ homogeneous solutions for both fluids I and II.

A noteworthy difference between the $O(\epsilon^2)$ interface boundary conditions and their $O(\epsilon)$ counterparts is that, as a result of the particular solution being evaluated at the interface, the $O(\epsilon^2)$ interface boundary conditions contain terms that grow with x. This makes sense physically because the amplitude of the particular solution grows with the distance propagated. Since the particular solution strikes the interface at oblique incidence, the distance propagated varies with position along the interface; consequently, its amplitude varies along the interface. It thus turns out that the $O(\epsilon^2)$ homogeneous solutions is composed of two parts: The part that accounts for the aforementioned particular solution has amplitudes which vary along its wavefront. The part that accounts for everything else has uniform amplitude along its wavefront. The propagation direction of both waves is assumed constant. The assumption of constant angle is difficult to maintain for all x and all t if the wave with the amplitude variation along the wavefront is overlooked.[3]

The introduction of some new notation makes matching the $O(\epsilon^2)$ interface boundary conditions more manageable. Recall that the $O(\epsilon^2)$ interface boundary conditions, given in Eqs. (5.34) and (5.36), contain terms that account for the displacement of the interface, the variation of the normal to the interface, and the $O(\epsilon^2)$ pressure and particle velocity differences. Singling out the $O(\epsilon^2)$ pressure and particle velocity differences for a moment, we see that they may be expanded into their homogeneous and particular parts and written in terms of

[3]A wave with amplitude variation along the wavefront is a solution of the wave equation. For example, $p = A(y)f(x - c_0t)$ is a solution of the one-dimensional wave equation, $p_{tt} - c_0^2 p_{xx} = 0$, where the subscripts indicate differentiation.

the modified velocity potential Φ,

$$p^{\mathrm{I}}_{(2)}\Big|_{z=0} - p^{\mathrm{II}}_{(2)}\Big|_{z=0} = -\rho^{\mathrm{I}}_0 \left[\frac{\partial \Phi^{\mathrm{I}}_{(2)h}}{\partial t} - m\frac{\partial \Phi^{\mathrm{II}}_{(2)h}}{\partial t} + \frac{\partial \Phi^{\mathrm{I}}_{(2)p}}{\partial t} - m\frac{\partial \Phi^{\mathrm{II}}_{(2)p}}{\partial t} \right.$$
$$\left. + \frac{1}{4}\left(\nabla^2 + \frac{1}{(c^{\mathrm{I}}_0)^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\mathrm{I}}_{(1)})^2 - \frac{m}{4}\left(\nabla^2 + \frac{1}{(c^{\mathrm{II}}_0)^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\mathrm{II}}_{(1)})^2\right]\Bigg|_{z=0} . \tag{5.94}$$

$$w^{\mathrm{I}}_{(2)}\Big|_{z=0} - w^{\mathrm{II}}_{(2)}\Big|_{z=0} = \left[\frac{\partial \Phi^{\mathrm{I}}_{(2)h}}{\partial z} - \frac{\partial \Phi^{\mathrm{II}}_{(2)h}}{\partial z} + \frac{\partial \Phi^{\mathrm{I}}_{(2)p}}{\partial z} - \frac{\partial \Phi^{\mathrm{II}}_{(2)p}}{\partial z} \right.$$
$$\left. + \frac{1}{2(c^{\mathrm{I}}_0)^2}\frac{\partial}{\partial t}\frac{\partial}{\partial z}(\Phi^{\mathrm{I}}_{(1)})^2 - \frac{1}{2(c^{\mathrm{II}}_0)^2}\frac{\partial}{\partial t}\frac{\partial}{\partial z}(\Phi^{\mathrm{II}}_{(1)})^2\right]\Bigg|_{z=0} . \tag{5.95}$$

Use of Eqs. (5.94) and (5.95) in Eqs. (5.34) and (5.36) leads to some lengthy expressions. We therefore introduce the following shorthand notation for not only the terms in Eqs. (5.94) and (5.95), but also for the terms in Eqs. (5.34) and (5.36) that represent the variation of the normal and the displacement of the interface:

A accounts for the particular solution of the particle velocity and is subdivided into two components, A_x and A_{reg}, where A_x represents those terms that, when evaluated at the interface, have x as a coefficient, and A_{reg} represents the 'regular terms' that do not have x as a coefficient,

B accounts for the nonlinear relation between the particle velocity and the modified velocity potential,

C accounts for the particular solution of the pressure and is also subdivided into two components, C_x and C_{reg}, for the same reasons as for A_x and A_{reg},

D accounts for the nonlinear relation between the pressure and the modified velocity potential,

E accounts for the displacement of the interface in the particle velocity boundary condition,

F accounts for the displacement of the interface in the pressure boundary condition,

G accounts for variation of the normal in the particle velocity boundary condition.

Mathematically, the shorthand notation is defined as follows:

$$A \equiv A_{\text{reg}} + xA_x \equiv \left.\frac{\partial \Phi^{\text{I}}_{(2)p}}{\partial z}\right|_{z=0} - \left.\frac{\partial \Phi^{\text{II}}_{(2)p}}{\partial z}\right|_{z=0} \quad , \tag{5.96}$$

$$B \equiv \left.\frac{1}{2(c^{\text{I}}_0)^2}\frac{\partial}{\partial t}\frac{\partial}{\partial z}(\Phi^{\text{I}}_{(1)})^2\right|_{z=0} - \left.\frac{1}{2(c^{\text{II}}_0)^2}\frac{\partial}{\partial t}\frac{\partial}{\partial z}(\Phi^{\text{II}}_{(1)})^2\right|_{z=0} \quad , \tag{5.97}$$

$$C \equiv C_{\text{reg}} + xC_x \equiv \left.\frac{\partial \Phi^{\text{I}}_{(2)p}}{\partial t}\right|_{z=0} - m\left.\frac{\partial \Phi^{\text{II}}_{(2)p}}{\partial t}\right|_{z=0} \quad , \tag{5.98}$$

$$D \equiv \frac{1}{4}\left(\nabla^2 + \frac{1}{(c^{\text{I}}_0)^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\text{I}}_{(1)})^2\Big|_{z=0} - \frac{m}{4}\left(\nabla^2 + \frac{1}{(c^{\text{II}}_0)^2}\frac{\partial^2}{\partial t^2}\right)(\Phi^{\text{II}}_{(1)})^2\Big|_{z=0} \quad , \tag{5.99}$$

$$E \equiv \left(\frac{\partial w^{\text{I}}_{(1)}}{\partial z} - \frac{\partial w^{\text{II}}_{(1)}}{\partial z}\right)\int w^{\text{II}}_{(1)}\,dt\Big|_{z=0} \quad , \tag{5.100}$$

$$F \equiv \frac{1}{\rho^{\text{I}}_0}\left(\frac{\partial p^{\text{I}}_{(1)}}{\partial z} - \frac{\partial p^{\text{II}}_{(1)}}{\partial z}\right)\int w^{\text{II}}_{(1)}\,dt\Big|_{z=0} \quad , \tag{5.101}$$

$$G \equiv (u^{\text{II}}_{(1)} - u^{\text{I}}_{(1)})\int \frac{\partial w^{\text{II}}_{(1)}}{\partial x}\,dt\Big|_{z=0} \quad . \tag{5.102}$$

Use of the foregoing definitions in the interface boundary conditions yields the following more compact form of the interface boundary conditions:

$$\left.\frac{\partial \Phi^{\text{I}}_{(2)h}}{\partial t}\right|_{z=0} - m\left.\frac{\partial \Phi^{\text{II}}_{(2)h}}{\partial t}\right|_{z=0} = -(C_{\text{reg}} + xC_x + D) + F \quad , \tag{5.103}$$

$$\left.\frac{\partial \Phi^{\text{I}}_{(2)h}}{\partial z}\right|_{z=0} - \left.\frac{\partial \Phi^{\text{II}}_{(2)h}}{\partial z}\right|_{z=0} = -(A_{\text{reg}} + xA_x + B + E + G) \quad . \tag{5.104}$$

Explicitly shown in Eqs. (5.103) and (5.104) is the interrelation between the $O(\epsilon^2)$ homogeneous solutions for fluids I and II.

An examination of the definitions of A through G reveals that they are quadratic forms of $O(\epsilon)$ terms. Moreover, A through G are defined at $z = 0$, and relations between the $O(\epsilon)$ terms at $z = 0$ are also known. It is thus possible to specify A through G in terms of the $O(\epsilon)$ incident signal, the properties of fluid I, and the nondimensional ratios of the fluid properties. The general procedure

is as follows: First, we use Eqs. (5.72) and (5.73) to express the functions $\Phi_{(1)}^{\mathrm{refl}}$ and $\Phi_{(1)}^{\mathrm{trans}}$ in terms of $\Phi_{(1)}^{\mathrm{inc}}$. Next, Snell's law and the law of specular reflection, Eqs. (5.52) and (5.54), are used to express the coefficients $\sin\theta_{(1)}^{\mathrm{refl}}$ and $\sin\theta_{(1)}^{\mathrm{trans}}$ in terms of $\sin\theta^{\mathrm{inc}}$. Using Eq. (5.65), we rewrite the coefficient $\cos\theta_{(1)}^{\mathrm{refl}}$ in terms of $\cos\theta^{\mathrm{inc}}$; the coefficient $\cos\theta_{(1)}^{\mathrm{trans}}$, which usually appears with the coefficient T, is eliminated in favor of $\cos\theta^{\mathrm{inc}}$ by way of Eq. (5.63). In general, we eliminate T in favor of R using Eqs. (5.63) and (5.64), or a combination thereof. Also note that the time and space derivatives of $\Phi_{(1)}$ are related by Eq. (5.50). The resulting expressions for A through G (with the 'x-dependent' parts of A and C separated from the 'regular' parts) are

$$A_{\mathrm{reg}}(\tau_\theta) = -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}(1-R^2)\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0}, \tag{5.105}$$

$$xA_x(\tau_\theta) = \frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}(1-R^2)\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)\frac{x\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0}, \tag{5.106}$$

$$B(\tau_\theta) = -\frac{1}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}(1-R^2)\left(1-\frac{n^2}{m}\right)\frac{\partial^2}{\partial t^2}(\Phi_{(1)}^{\mathrm{inc}})^2\Bigg|_{z=0}, \tag{5.107}$$

$$C_{\mathrm{reg}}(\tau_\theta) = -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\frac{R}{\cos^2\theta^{\mathrm{inc}}}\frac{\partial^2}{\partial t^2}(\Phi_{(1)}^{\mathrm{inc}})^2\Bigg|_{z=0}, \tag{5.108}$$

$$xC_x(\tau_\theta) = -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\left[(1+R)^2\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)-2R\right]\frac{x\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0}, \tag{5.109}$$

$$D(\tau_\theta) = \frac{1}{(c_0^{\mathrm{I}})^2}\left[\frac{(1+R)^2}{2}\left(1-\frac{n^2}{m}\right)\frac{\partial^2}{\partial t^2}(\Phi_{(1)}^{\mathrm{inc}})^2-2R\cos^2\theta^{\mathrm{inc}}\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\right]\Bigg|_{z=0}, \tag{5.110}$$

$$\begin{aligned}E(\tau_\theta) = {} & \frac{1}{(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}(1-R^2) \\ & \times\left[\sin^2\theta^{\mathrm{inc}}\left(\frac{m-1}{m}\right)-\left(1-\frac{n^2}{m}\right)\right]\Phi_{(1)}^{\mathrm{inc}}\frac{\partial^2\Phi_{(1)}^{\mathrm{inc}}}{\partial t^2}\Bigg|_{z=0},\end{aligned} \tag{5.111}$$

$$F(\tau_\theta) = \frac{\cos^2\theta^{\mathrm{inc}}}{(c_0^{\mathrm{I}})^2}(1-R)^2(m-1)\Phi_{(1)}^{\mathrm{inc}}\frac{\partial^2\Phi_{(1)}^{\mathrm{inc}}}{\partial t^2}\Bigg|_{z=0}, \tag{5.112}$$

$$G(\tau_\theta) = \frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\sin^2\theta^{\mathrm{inc}}}{(c_0^{\mathrm{I}})^2}(1 - R^2)\left(\frac{m-1}{m}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0} \quad . \tag{5.113}$$

Equations (5.105)-(5.113) are a significant result because, as we soon find out, the terms A through G appear throughout the homogeneous solutions. Obtaining simplified forms of the homogenous solutions for special cases such as reflection from a rigid wall ($R = 1$) or reflection from a pressure release surface ($R = -1$) is now relatively straightforward. (In the next chapter, we analyze the special case of intromission, $R = 0$.) Moreover, the dependence of the terms on the changes in physical properties of the fluids is now apparent, specifically,

$$1 - \frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m} = \frac{\beta^{\mathrm{II}}}{\rho_0^{\mathrm{II}}(c_0^{\mathrm{II}})^2}\left[\!\!\left[\frac{\rho_0 c_0^2}{\beta}\right]\!\!\right] \quad , \tag{5.114}$$

$$1 - \frac{n^2}{m} = \frac{1}{\rho_0^{\mathrm{II}}(c_0^{\mathrm{II}})^2}\left[\!\!\left[\rho_0 c_0^2\right]\!\!\right] \quad , \tag{5.115}$$

$$\frac{m-1}{m} = \frac{1}{\rho_0^{\mathrm{II}}}[\![\rho_0]\!] \quad , \tag{5.116}$$

where the symbol $[\![\ ]\!]$ indicates the jump of the enclosed quantity at the interface, that is, (property$^{\mathrm{II}}$ – property$^{\mathrm{I}}$). Equation (5.114) represents the jump in a new property, the ratio of the compressibility to the coefficient of nonlinearity, whereas Eqs (5.115) and (5.116) represent the jump in the compressibility and the density, respectively. Note that the terms that represent the motion of the interface vanish if both the densities and small-signal sound speeds are matched. Moreover, it turns out that the general condition for no reflections (perfect transmission) is that the densities, the small-signal sound speeds, *and* the coefficients of nonlinearity be matched.

We now solve the $O(\epsilon^2)$ homogeneous wave equations, Eqs. (5.87) and (5.93), subject to the $O(\epsilon^2)$ interface boundary conditions as given in Eqs. (5.103) and (5.104). We earlier noted that the $O(\epsilon^2)$ interface boundary conditions contain terms that grow with x. Futhermore, we noted that these terms are accounted for by a separate function—one that exhibits an x-dependence when evaluated at the interface. This function, which is denoted by a subscript a, and a constant amplitude function, which is denoted by the subscript b, satisfy the interface boundary conditions. The reflected and transmitted signals are sought in the following form:

$$\Phi^{\mathrm{I}}_{(2)h}(\mathbf{r},t) = \Phi^{\mathrm{inc}}_{(2)}(\tau^{\mathrm{inc}}) + (\mathbf{n}^{\mathrm{refl}}_{(2)a\perp}\cdot\mathbf{r})\Phi^{\mathrm{refl}}_{(2)a}(\tau^{\mathrm{refl}}_{(2)a}) + \Phi^{\mathrm{refl}}_{(2)b}(\tau^{\mathrm{refl}}_{(2)b}) \quad , \tag{5.117}$$

$$\Phi^{\mathrm{II}}_{(2)h}(\mathbf{r},t) = (\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}\cdot\mathbf{r})\Phi^{\mathrm{trans}}_{(2)a}(\tau^{\mathrm{trans}}_{(2)a}) + \Phi^{\mathrm{trans}}_{(2)b}(\tau^{\mathrm{trans}}_{(2)b}) \quad , \tag{5.118}$$

where the amplitude variation along the interface is given by $(\mathbf{n}^{\mathrm{refl}}_{(2)a\perp}\cdot\mathbf{r})$ and $(\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}\cdot\mathbf{r})$ and where

$$\tau^{\mathrm{refl}}_{(2)a} = t - \frac{\mathbf{n}^{\mathrm{refl}}_{(2)a}\cdot\mathbf{r}}{c_0^{\mathrm{I}}} \quad , \tag{5.119}$$

$$\tau^{\mathrm{refl}}_{(2)b} = t - \frac{\mathbf{n}^{\mathrm{refl}}_{(2)b}\cdot\mathbf{r}}{c_0^{\mathrm{I}}} \quad , \tag{5.120}$$

$$\tau^{\mathrm{trans}}_{(2)a} = t - \frac{\mathbf{n}^{\mathrm{trans}}_{(2)a}\cdot\mathbf{r}}{c_0^{\mathrm{II}}} \quad , \tag{5.121}$$

$$\tau^{\mathrm{trans}}_{(2)b} = t - \frac{\mathbf{n}^{\mathrm{trans}}_{(2)b}\cdot\mathbf{r}}{c_0^{\mathrm{II}}} \quad , \tag{5.122}$$

$$\mathbf{n}^{\mathrm{refl}}_{(2)a} = \mathbf{i}\sin\theta^{\mathrm{refl}}_{(2)a} + \mathbf{k}\cos\theta^{\mathrm{refl}}_{(2)a} \quad , \tag{5.123}$$

$$\mathbf{n}^{\mathrm{refl}}_{(2)b} = \mathbf{i}\sin\theta^{\mathrm{refl}}_{(2)b} + \mathbf{k}\cos\theta^{\mathrm{refl}}_{2(b)} \quad , \tag{5.124}$$

$$\mathbf{n}^{\mathrm{trans}}_{(2)a} = \mathbf{i}\sin\theta^{\mathrm{trans}}_{(2)a} + \mathbf{k}\cos\theta^{\mathrm{trans}}_{(2)a} \quad , \tag{5.125}$$

$$\mathbf{n}^{\mathrm{trans}}_{(2)b} = \mathbf{i}\sin\theta^{\mathrm{trans}}_{(2)b} + \mathbf{k}\cos\theta^{\mathrm{trans}}_{(2)b} \quad , \tag{5.126}$$

$$\mathbf{n}^{\mathrm{refl}}_{(2)a\perp} = \mathbf{i}\sin\theta^{\mathrm{refl}}_{(2)a\perp} + \mathbf{k}\cos\theta^{\mathrm{refl}}_{(2)a\perp} \quad , \tag{5.127}$$

$$\mathbf{n}^{\mathrm{trans}}_{(2)a\perp} = \mathbf{i}\sin\theta^{\mathrm{refl}}_{(2)a\perp} + \mathbf{k}\sin\theta^{\mathrm{refl}}_{(2)a\perp} \quad . \tag{5.128}$$

That $(\mathbf{n}^{\mathrm{refl}}_{(2)a\perp}\cdot\mathbf{r})\Phi^{\mathrm{refl}}_{(2)a}(\tau^{\mathrm{refl}}_{(2)a})$ is a solution of the homogeneous wave equation may be verified by direct substitution if it is noted that the vectors $\mathbf{n}^{\mathrm{refl}}_{(2)a\perp}$ and $\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}$ are perpendicular to the direction of propagation, that is,

$$\mathbf{n}^{\mathrm{refl}}_{(2)a}\cdot\mathbf{n}^{\mathrm{refl}}_{(2)a\perp} = 0 \quad , \tag{5.129}$$

$$\mathbf{n}^{\mathrm{trans}}_{(2)a}\cdot\mathbf{n}^{\mathrm{trans}}_{(2)a\perp} = 0 \quad . \tag{5.130}$$

However, two possibilities exist for both $\mathbf{n}^{\mathrm{refl}}_{(2)a\perp}$ and $\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}$: it may have a component in either the $+\mathbf{i}$ direction or the $-\mathbf{i}$ direction. We choose both $\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}$ and $\mathbf{n}^{\mathrm{ref}}_{(2)a\perp}$ to have components in the $+\mathbf{i}$ direction; see Fig. 5.3. Direct substitution of Eqs. (5.117) and (5.118) into the $O(\epsilon^2)$ interface boundary conditions, Eqs. (5.103) and (5.104), yields what appears to be a system of two equations and four unknowns with four angles to be determined subject to the two conditions that the solution be valid for all x and all t. Fortunately, we may separate the terms that exhibit growth with x and demand that they equate separately. We

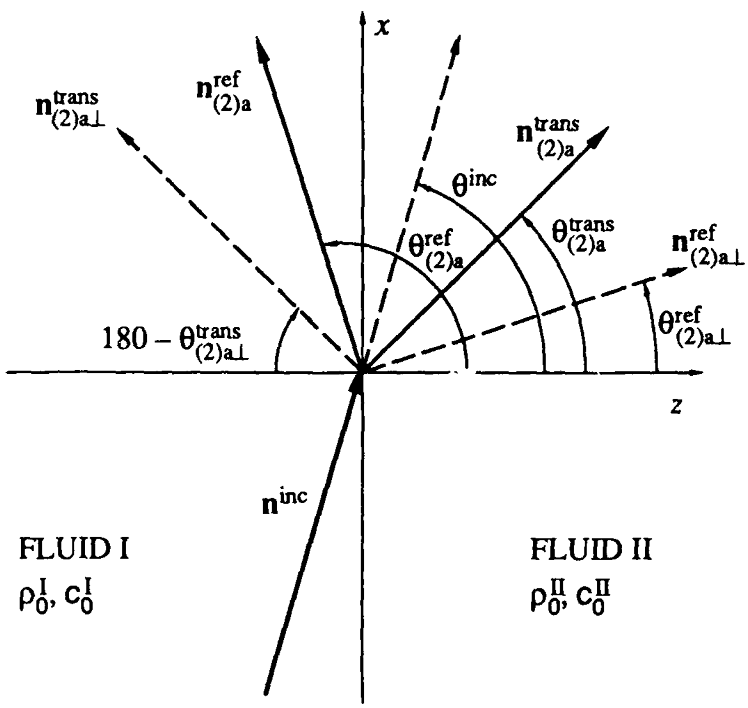

Figure 5.3 DIAGRAM SHOWING DIRECTION OF $\mathbf{n}^{refl}_{(2)a\perp}$ AND $\mathbf{n}^{trans}_{(2)a\perp}$.

are thus left with *two* systems of two equations and two unknowns with two angles to be determined subject to the two conditions that the solutions be valid for all x and all t—one system for the 'x-dependent terms' and one for the 'regular terms.' The system for the 'x-dependent terms' is

$$-\frac{\cos\theta^{\text{refl}}_{(2)a}}{c_0^{\text{I}}}\sin\theta^{\text{refl}}_{(2)a\perp}\left.\frac{\partial\Phi^{\text{refl}}_{(2)a}}{\partial t}\right|_{z=0}+\frac{\cos\theta^{\text{trans}}_{(2)a}}{c_0^{\text{II}}}\sin\theta^{\text{trans}}_{(2)a\perp}\left.\frac{\partial\Phi^{\text{trans}}_{(2)a}}{\partial t}\right|_{z=0}=-A_x \quad , \tag{5.131}$$

$$\sin\theta^{\text{refl}}_{(2)a\perp}\left.\frac{\partial\Phi^{\text{refl}}_{(2)a}}{\partial t}\right|_{z=0}-m\sin\theta^{\text{trans}}_{(2)a\perp}\left.\frac{\partial\Phi^{\text{trans}}_{(2)a}}{\partial t}\right|_{z=0}=-C_x \quad . \tag{5.132}$$

The system for the 'regular terms' is

$$\begin{aligned}&-\frac{\cos\theta^{\text{inc}}}{c_0^{\text{I}}}\left.\frac{\partial\Phi^{\text{inc}}_{(2)}}{\partial t}\right|_{z=0}-\frac{\cos\theta^{\text{refl}}_{(2)b}}{c_0^{\text{I}}}\left.\frac{\partial\Phi^{\text{refl}}_{(2)b}}{\partial t}\right|_{z=0}+\cos\theta^{\text{refl}}_{(2)a\perp}\left.\Phi^{\text{refl}}_{(2)a}\right|_{z=0}\\&-\cos\theta^{\text{trans}}_{(2)a\perp}\left.\Phi^{\text{trans}}_{(2)a}\right|_{z=0}+\frac{\cos\theta^{\text{trans}}_{(2)b}}{c_0^{\text{II}}}\left.\frac{\partial\Phi^{\text{trans}}_{(2)b}}{\partial t}\right|_{z=0}=-(A_{\text{reg}}+B+E+G) \quad ,\end{aligned} \tag{5.133}$$

$$\left.\frac{\partial\Phi^{\text{inc}}_{(2)}}{\partial t}\right|_{z=0}+\left.\frac{\partial\Phi^{\text{refl}}_{(2)b}}{\partial t}\right|_{z=0}-m\left.\frac{\partial\Phi^{\text{trans}}_{(2)b}}{\partial t}\right|_{z=0}=-(C_{\text{reg}}+D)+F \quad . \tag{5.134}$$

Since the unknowns $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)a}$ appear in the 'regular term' system, it is required that 'x-dependent term' system be solved first. To solve the 'x-dependent term' system, we first find the angles by applying the conditions that the solutions be valid for all x and all t. This is accomplished in the same way as it was in the $O(\epsilon)$ case—the x and t derivatives of Eqs. (5.131) and (5.132) are taken. The resulting sets of equations are then forced to be linearly dependent thereby making their solutions identical. To make the x and t derivatives of Eqs. (5.131) and (5.132) linearly dependent, the following two conditions must be true:

$$\frac{\sin\theta^{\text{refl}}_{(2)a}}{c_0^{\text{I}}}=\frac{\sin\theta^{\text{inc}}}{c_0^{\text{I}}} \quad , \tag{5.135}$$

$$\frac{\sin\theta^{\text{trans}}_{(2)a}}{c_0^{\text{II}}}=\frac{\sin\theta^{\text{inc}}}{c_0^{\text{I}}} \quad . \tag{5.136}$$

Accordingly, we see that

$$\theta^{\text{refl}}_{(2)a}=\theta^{\text{refl}}_{(1)}=180^\circ-\theta^{\text{inc}} \tag{5.137}$$

and

$$\theta^{\text{trans}}_{(2)a} = \theta^{\text{trans}}_{(1)} \quad . \tag{5.138}$$

Thus, the signals that match the particular solution at the interface propagate in the same direction as the $O(\epsilon)$ signal. That is to say, these $O(\epsilon^2)$ homogeneous solutions obey Snell's law and the law of specular reflection. Since the angles $\theta^{\text{refl}}_{(2)a}$ and $\theta^{\text{trans}}_{(2)a}$ are known, the angles $\theta^{\text{refl}}_{(2)a\perp}$ and $\theta^{\text{trans}}_{(2)a\perp}$ are also known,

$$\theta^{\text{refl}}_{(2)a\perp} = \theta^{\text{refl}}_{(1)} - 90^\circ = 90^\circ - \theta^{\text{inc}} \quad , \tag{5.139}$$

$$\theta^{\text{trans}}_{(2)a\perp} = \theta^{\text{trans}}_{(1)} + 90^\circ \quad . \tag{5.140}$$

Equations (5.131) and (5.132) may now be solved for $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)a}$. Use of Eqs. (5.137)-(5.140) in rearranged forms of Eqs. (5.131) and (5.132) yields

$$\left.\frac{\partial\Phi^{\text{refl}}_{(2)a}}{\partial t}\right|_{z=0} = -\frac{1}{2\cos\theta^{\text{inc}}}\left((1-R)C_x + \frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}(1+R)A_x\right) \quad , \tag{5.141}$$

$$\left.\frac{\partial\Phi^{\text{trans}}_{(2)a}}{\partial t}\right|_{z=0} = \frac{n}{m^2}\frac{1}{2\cos\theta^{\text{inc}}}\frac{(1+R)^2}{1-R}\left(C_x - \frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}A_x\right) \quad , \tag{5.142}$$

where we have made use of the following:

$$\frac{Z^{\text{I}}}{Z^{\text{I}} + Z^{\text{II}}} = \frac{1-R}{2} \quad , \tag{5.143}$$

$$\frac{1}{\rho_0^{\text{I}}}\frac{Z^{\text{II}}Z^{\text{I}}}{Z^{\text{I}} + Z^{\text{II}}} = -\frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}\frac{1+R}{2} \quad , \tag{5.144}$$

$$\frac{1}{m}\frac{Z^{\text{II}}}{Z^{\text{I}} + Z^{\text{II}}} = \frac{1+R}{2m} \quad , \tag{5.145}$$

$$\frac{1}{\rho_0^{\text{II}}}\frac{Z^{\text{I}}Z^{\text{II}}}{Z^{\text{I}} + Z^{\text{II}}} = \frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}\frac{1+R}{2m} \quad , \tag{5.146}$$

$$\frac{1}{\cos\theta^{\text{trans}}_{(1)}} = \frac{1}{\cos\theta^{\text{inc}}}\frac{n}{m}\left(\frac{1+R}{1-R}\right) \quad . \tag{5.147}$$

Equations (5.141) and (5.142) may be integrated once with respect to time noting that the integration constant must be zero in order to satisfy quiet conditions, that is, in the absence of an incident signal, a reflected or transmitted signal does not exist. Since both A_x and C_x are given in terms of the time derivative of the incident signal [see Eqs. (5.106) and (5.109)], integration is possible in closed

form. That this integration is possible is important because the expressions $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)a}$ show up explicitly in the $O(\epsilon^2)$ form of the particle velocity. To obtain expressions for $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)b}$ away from the interface, we replace the current independent variable τ_θ with $\tau^{\text{refl}}_{(1)}$ for the $O(\epsilon^2)$ reflected field and with $\tau^{\text{trans}}_{(1)}$ for the $O(\epsilon^2)$ transmitted field. Thus, the expressions for $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)a}$ are

$$\Phi^{\text{refl}}_{(2)a}(\tau^{\text{refl}}_{(1)}) = -\frac{1}{2\cos\theta^{\text{inc}}}\left((1-R)\int C_x(\tau^{\text{refl}}_{(1)})\,dt + \frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}(1-R)\int A_x(\tau^{\text{refl}}_{(1)})\,dt\right) \tag{5.148}$$

and

$$\Phi^{\text{trans}}_{(2)a}(\tau^{\text{trans}}_{(1)}) = \frac{n}{m^2}\frac{1}{2\cos\theta^{\text{inc}}}\frac{(1+R)^2}{1-R} \times\left(\int C_x(\tau^{\text{trans}}_{(1)})\,dt - \frac{c_0^{\text{I}}}{\cos\theta^{\text{inc}}}\int A_x(\tau^{\text{trans}}_{(1)})\,dt\right). \tag{5.149}$$

The system composed of 'regular terms' is now solved in an identical fashion: First, the x and t derivatives of the pair of equations are taken, and the two conditions that force the x and t derivatives of the equations to be linearly dependent are obtained. The two conditions are

$$\frac{\sin\theta^{\text{refl}}_{(2)b}}{c_0^{\text{I}}} = \frac{\sin\theta^{\text{inc}}}{c_0^{\text{I}}} \quad , \tag{5.150}$$

$$\frac{\sin\theta^{\text{trans}}_{(2)b}}{c_0^{\text{II}}} = \frac{\sin\theta^{\text{inc}}}{c_0^{\text{I}}} \quad . \tag{5.151}$$

Thus, the angles at which the $O(\epsilon^2)$ homogeneous solutions leave the interface are the same as their $O(\epsilon)$ counterparts,

$$\theta^{\text{refl}}_{(2)b} = \theta^{\text{refl}}_{(1)} = 180^\circ - \theta^{\text{inc}} \quad , \tag{5.152}$$

$$\theta^{\text{trans}}_{(2)b} = \theta^{\text{trans}}_{(1)} \quad . \tag{5.153}$$

That is to say, these $O(\epsilon^2)$ homogenous solutions also obey Snell's law and the law of specular reflection. Using Eqs. (5.152) and (5.153) in the 'regular term' system [Eqs. (5.133) and (5.134)] and then rearranging leads to the following expression for $\frac{\partial}{\partial t}\Phi^{\text{refl}}_{(2)b}$ and $\frac{\partial}{\partial t}\Phi^{\text{trans}}_{(2)b}$ at $z = 0$:

$$\left.\frac{\partial \Phi^{\mathrm{refl}}_{(2)b}}{\partial t}\right|_{z=0} = R \left.\frac{\partial \Phi^{\mathrm{inc}}_{(2)}}{\partial t}\right|_{z=0} - \frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}\left(\frac{1+R}{2}\right)\sin\theta^{\mathrm{inc}}\left.\left(\Phi^{\mathrm{refl}}_{(2)a} + \frac{\Phi^{\mathrm{trans}}_{(2)a}}{n}\right)\right|_{z=0}$$
$$- \frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}\left(\frac{1+R}{2}\right)(A_{\mathrm{reg}} + B + E + G)$$
$$- \left(\frac{1-R}{2}\right)(C_{\mathrm{reg}} + D - F) \quad , \tag{5.154}$$

$$\left.\frac{\partial \Phi^{\mathrm{trans}}_{(2)b}}{\partial t}\right|_{z=0} = \frac{T}{m}\left.\frac{\partial \Phi^{\mathrm{inc}}_{(2)}}{\partial t}\right|_{z=0} - \frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}\left(\frac{1+R}{2}\right)\frac{\sin\theta^{\mathrm{inc}}}{m}\left.\left(\Phi^{\mathrm{refl}}_{(2)a} + \frac{\Phi^{\mathrm{trans}}_{(2)a}}{n}\right)\right|_{z=0}$$
$$- \frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}\left(\frac{1+R}{2}\right)\frac{1}{m}(A_{\mathrm{reg}} + B + E + G)$$
$$+ \left(\frac{1+R}{2}\right)\frac{1}{m}(C_{\mathrm{reg}} + D - F) \quad , \tag{5.155}$$

where we have again made use of Eqs. (5.143)–(5.147). Integration of Eqs. (5.154) and (5.155) is, in principle, possible, but not in closed form. This is not a problem, however, because $\Phi^{\mathrm{refl}}_{(2)b}$ and $\Phi^{\mathrm{trans}}_{(2)b}$, unlike $\Phi^{\mathrm{refl}}_{(2)a}$ and $\Phi^{\mathrm{trans}}_{(2)a}$, do not appear explictly in any measurable quantity, only their derivatives. To obtain expressions for $\frac{\partial}{\partial t}\,\Phi^{\mathrm{refl}}_{(2)b}$ and $\frac{\partial}{\partial t}\,\Phi^{\mathrm{trans}}_{(2)b}$ away from the interface, we replace the independent variable τ_θ with $\tau^{\mathrm{refl}}_{(1)}$ for the reflected field and with $\tau^{\mathrm{trans}}_{(1)}$ for the transmitted field.

In summary, although we have not obtained closed form solutions for $\Phi^{\mathrm{I}}_{(2)h}$ and $\Phi^{\mathrm{II}}_{(2)h}$, we have obtained closed form solutions for the derivatives. This is significant because only the derivatives of $\Phi^{\mathrm{I}}_{(2)}$ appear in measurable quantities such as the pressure or particle velocity. The derivatives of $\Phi^{\mathrm{I}}_{(2)h}$ and $\Phi^{\mathrm{II}}_{(2)h}$ are

$$\frac{\partial}{\partial t}\,\Phi^{\mathrm{I}}_{(2)h}(\mathbf{r},t) = \frac{\partial \Phi^{\mathrm{inc}}_{(2)}}{\partial t} + (\mathbf{n}^{\mathrm{ref}}_{(1)\perp}\cdot\mathbf{r})\,\frac{\partial \Phi^{\mathrm{refl}}_{(2)a}}{\partial t} + \frac{\partial \Phi^{\mathrm{refl}}_{(2)b}}{\partial t} \quad , \tag{5.156}$$

$$\nabla\Phi^{\mathrm{I}}_{(2)h}(\mathbf{r},t) = -\frac{\mathbf{n}^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\,\frac{\partial \Phi^{\mathrm{inc}}_{(2)}}{\partial t} + \mathbf{n}^{\mathrm{ref}}_{(1)\perp}\,\Phi^{\mathrm{refl}}_{(2)a} - \frac{\mathbf{n}^{\mathrm{ref}}_{(1)}}{c_0^{\mathrm{I}}}(\mathbf{n}^{\mathrm{ref}}_{(1)\perp}\cdot\mathbf{r})\,\frac{\partial \Phi^{\mathrm{refl}}_{(2)a}}{\partial t} - \frac{\mathbf{n}^{\mathrm{ref}}_{(1)}}{c_0^{\mathrm{I}}}\,\frac{\partial \Phi^{\mathrm{refl}}_{(2)b}}{\partial t} \quad , \tag{5.157}$$

$$\frac{\partial}{\partial t}\,\Phi^{\mathrm{II}}_{(2)h}(\mathbf{r},t) = (\mathbf{n}^{\mathrm{trans}}_{(1)\perp}\cdot\mathbf{r})\,\frac{\partial \Phi^{\mathrm{trans}}_{(2)a}}{\partial t} + \frac{\partial \Phi^{\mathrm{trans}}_{(2)b}}{\partial t} \quad , \tag{5.158}$$

$$\nabla\Phi^{\mathrm{II}}_{(2)h}(\mathbf{r},t) = \mathbf{n}^{\mathrm{trans}}_{(1)\perp}\,\Phi^{\mathrm{trans}}_{(2)a} - \frac{\mathbf{n}^{\mathrm{trans}}_{(1)}}{c_0^{\mathrm{I}}}(\mathbf{n}^{\mathrm{trans}}_{(2)a\perp}\cdot\mathbf{r})\,\frac{\partial \Phi^{\mathrm{trans}}_{(2)a}}{\partial t} - \frac{\mathbf{n}^{\mathrm{trans}}_{(1)}}{c_0^{\mathrm{I}}}\,\frac{\partial \Phi^{\mathrm{trans}}_{(2)b}}{\partial t} \quad , \tag{5.159}$$

where $\mathbf{n}^{\text{ref}}_{(1)\perp}$ and $\mathbf{n}^{\text{trans}}_{(1)\perp}$ are given by

$$\mathbf{n}^{\text{ref}}_{(1)\perp} = \mathbf{i}\cos\theta^{\text{inc}} + \mathbf{k}\sin\theta^{\text{inc}} \quad , \tag{5.160}$$

$$\mathbf{n}^{\text{trans}}_{(1)\perp} = \mathbf{i}\cos\theta^{\text{trans}}_{(1)} - \mathbf{k}\sin\theta^{\text{trans}}_{(1)} \quad . \tag{5.161}$$

All the components of Eqs. (5.156)–(5.159) are known: $\Phi^{\text{refl}}_{(2)a}$ and $\Phi^{\text{trans}}_{(2)a}$ are given by Eqs. (5.148) and (5.149), and their derivatives, $\frac{\partial}{\partial t}\Phi^{\text{refl}}_{(2)a}$ and $\frac{\partial}{\partial t}\Phi^{\text{trans}}_{(2)a}$, are readily obtained. The functions $\frac{\partial}{\partial t}\Phi^{\text{refl}}_{(2)b}$ and $\frac{\partial}{\partial t}\Phi^{\text{trans}}_{(2)b}$ are given by Eqs. (5.154) and (5.155). It is also noted that the $O(\epsilon^2)$ homogeneous solutions propagate in directions predicted by Snell's law and the law of specular reflection.

Matching the $O(\epsilon^2)$ source condition

The $O(\epsilon^2)$ boundary condition at the source is now matched to the $O(\epsilon^2)$ solution evaluated at the source, $\mathbf{r} = \mathbf{r}_0$. As we noted when matching the $O(\epsilon)$ boundary condition at the source, the reflected signal is not involved in matching the boundary condition at the source. The $O(\epsilon^2)$ boundary condition at the source is given in Eq. (5.40). The relation between $\mathbf{u}_{(2)}$ and the modified velocity potential $\Phi_{(2)}$ is given in Eq. (5.30). Use of Eq. (5.40) and the general solution for $\Phi^{\text{I}}_{(2)}$, Eq. (5.80), in Eq. (5.30) leads to

$$\nabla\Phi^{\text{inc}}_{(2)}\Big|_{\mathbf{r}=\mathbf{r}_0} = -\left.\nabla\Phi^{\text{I}}_{(2)p}\right|_{\mathbf{r}=\mathbf{r}_0} - \frac{1}{2(c^{\text{I}}_0)^2}\frac{\partial}{\partial t}\nabla(\Phi^{\text{inc}}_{(1)})^2\Big|_{\mathbf{r}=\mathbf{r}_0} - \frac{\mathbf{n}^{\text{inc}}}{c^{\text{I}}_0}\frac{\partial}{\partial t}S_{(2)}(t-\tau_0) \quad . \tag{5.162}$$

The first term on the right-hand side of Eq. (5.162) forces the $O(\epsilon^2)$ particular solution to vanish at the source. The physical interpretation is that no self-action of the incident wave has occurred when the wave is still at the source. The second term on the right-hand side accounts for the nonlinear relationship between the modified velocity potential and the particle velocity. As we see below, this term disappears if the equation is re-expressed in terms of a measurable quantity such as the pressure or particle velocity. The third term on the right-hand side accounts for the $O(\epsilon^2)$ signal at the source. Physically, an $O(\epsilon^2)$ signal at the source represents a local effect at the source such as the finite displacement of the source. This local effect must not be overlooked because it is of the same order as the local effects at the interface—the finite displacement of the interface and the variation of the normal to the interface. Evaluating $\nabla\Phi^{\text{I}}_{(2)p}$ (neglecting reflections) and rearranging leads to the following:

$$\left.\frac{\partial\Phi^{\rm inc}_{(2)}}{\partial t}\right|_{\mathbf{r}=\mathbf{r}_0} = -\frac{\beta^{\rm I}}{2(c_0^{\rm I})^2}\left[\left(\frac{\partial\Phi^{\rm inc}_{(1)}}{\partial t}\right)^2 - (\mathbf{n}^{\rm inc}\cdot\mathbf{r})\frac{\partial}{\partial t}\left(\frac{\partial\Phi^{\rm inc}_{(1)}}{\partial t}\right)\right]\Bigg|_{\mathbf{r}=\mathbf{r}_0} - \frac{1}{2(c_0^{\rm I})^2}\left.\frac{\partial^2}{\partial t^2}(\Phi^{\rm inc}_{(1)})^2\right|_{\mathbf{r}=\mathbf{r}_0} + \frac{\partial}{\partial t}S_{(2)}(t-\tau_0) \quad . \tag{5.163}$$

A closed form integral of Eq. (5.163) was not found. If the source condition is harmonic, however, the integral may be obtained. The form of Eq. (5.163) is nevertheless acceptable because all measurable quantities depend on derivatives of Φ and not on Φ itself. For example, we may use Eq. (5.26) to rearrange Eq. (5.163):

$$-\left.\frac{p^{\rm inc}_{(2)}}{\rho_0^{\rm I}}\right|_{\mathbf{r}=\mathbf{r}_0} = -\frac{\beta^{\rm I}}{2(c_0^{\rm I})^2}\left[\left(\frac{\partial\Phi^{\rm inc}_{(1)}}{\partial t}\right)^2 - (\mathbf{n}^{\rm inc}\cdot\mathbf{r})\frac{\partial}{\partial t}\left(\frac{\partial\Phi^{\rm inc}_{(1)}}{\partial t}\right)\right]\Bigg|_{\mathbf{r}=\mathbf{r}_0} + \frac{\partial}{\partial t}S_{(2)}(t-\tau_0) \quad . \tag{5.164}$$

We have thus found the expression for the $O(\epsilon^2)$ incident pressure in terms of the $O(\epsilon)$ incident pressure [related to $\partial\Phi^{\rm inc}_{(1)}/\partial t$ by Eq. (5.25)] and the $O(\epsilon^2)$ pressure at the source (related to $\partial S_{(2)}/\partial t$). Other such manipulations are possible.

Summary of $O(\epsilon^2)$ Results

Because closed form solutions have not been obtained for $\Phi^{\rm refl}_{(2)b}$ and $\Phi^{\rm trans}_{(2)b}$, we have no closed form solutions for $\Phi^{\rm I}_{(2)}$ and $\Phi^{\rm II}_{(2)}$. Solutions for any $O(\epsilon^2)$ measurable quantity may, however, be obtained because measurable quantities depend only on the derivatives of $\Phi^{\rm I}_{(2)}$ and $\Phi^{\rm II}_{(2)}$. The derivatives of the functions $\Phi^{\rm I}_{(2)p}$ and $\Phi^{\rm II}_{(2)p}$ may be obtained by differentiating Eqs. (5.92) and (5.86), respectively. The derivatives of $\Phi^{\rm I}_{(2)h}$ and $\Phi^{\rm II}_{(2)h}$ are given in Eqs. (5.156)-(5.159).

The major results of this section are as follows: (1) Closed form expressions have been obtained for the $O(\epsilon^2)$ particular solutions for fluids I and II, Eqs. (5.92) and (5.86), respectively. (2) The propagation directions of the growth terms in the $O(\epsilon^2)$ particular solutions for fluids I and II are given by Snell's law and the law of specular reflection. (3) Closed form expressions have been obtained for the derivatives of the $O(\epsilon^2)$ homogeneous solutions for fluids I and II, Eqs. (5.156)--(5.159), respectively. (4) The propagation direction of the $O(\epsilon^2)$ homogeneous solutions are given by Snell's law and the law of specular reflection. (5) Expressions have been obtained for the quantities A through G, which represent the motion of the interface, variation of the normal to the interface, and the

nonlinearity of the relations between the acoustic variables, Eqs. (5.105)—(5.113), respectively. All of the foregoing have been obtained assuming an arbitrary source function.

5–5 Complex Transmission Angle

In this section we briefly point out what steps should be taken if the incident angle is greater than the critical angle. Above critical incidence, the transmission angle and the retarded time τ^{trans} are complex. Moreover, the definitions of the reflection and transmission coefficients, Eqs. (5.57) and (5.58), are, strictly speaking, no longer valid because the reflected and transmitted signals are no longer in phase with the incident signal. The problem may be resolved by using complex notation. We start by assuming an incident harmonic wave, say $\Phi^{\text{inc}}_{(1)} = \text{Re}[e^{-i\omega t}\tilde{\Phi}^{\text{inc}}_{(1)}(x,z)]$, where $i \equiv \sqrt{-1}$ and $\tilde{\Phi}^{\text{inc}}_{(1)}(x,z)$ is a complex amplitude. All the time derivatives in linear terms are then replaced by $-i\omega$. To compute the quadratic terms which contribute to the second harmonic, we must do only a little more work. For example, to compute the time derivative of $(\partial\Phi_{(1)}/\partial t)^2$ which appears in the wave equation Eq. (5.2), we note first that

$$\left(\frac{\partial\Phi_{(1)}}{\partial t}\right)^2 = -\frac{\omega^2}{2}\text{Re}\left[e^{-2i\omega t}\,\tilde{\Phi}^2_{(1)} - \tilde{\Phi}_{(1)}\tilde{\Phi}^*_{(1)}\right] \quad ,$$

where $\tilde{\Phi}^*_{(1)}$ is the complex conjugate of $\tilde{\Phi}_{(1)}$. The time derivative of $(\partial\Phi_{(1)}/\partial t)^2$ is, accordingly, given by

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}}{\partial t}\right)^2 = -\frac{\omega^2}{2}\text{Re}\left[-2i\omega\, e^{-2i\omega t}\,\tilde{\Phi}^2_{(1)}\right] \quad .$$

Thus, everything we have done may be applied to incident angles above the critical angle provided that (1) the time derivatives [illegible]r terms are replaced by $-i\omega$, (2) the time derivatives of quadratic ter[illegible] replaced by $-2i\omega$, and (3) the amplitude of the quadratic terms is multiplied by $\frac{1}{2}$.

The more general case of an incident pulse may also be treated using complex notation. The incident pulse is considered as a superposition of harmonic waves. Care must exercised in this case, however, because each harmonic interacts not only with itself (as above), but also with every other harmonic. Thus, it turns out that the $O(\epsilon^2)$ solution contains convolution integrals of the Fourier transform of the pulse with itself.

5–6 Summary

In this chapter, we have analyzed the reflection and refraction of finite-amplitude plane waves that are obliquely incident on an initially plane fluid-fluid interface. The procedure used is second-order perturbation analysis; the source boundary condition was arbitrary. Specific terms that account for the displacement of the interface, the variation of the normal to the interface, and the $O(\epsilon^2)$ source condition are identified. It is found that the $O(\epsilon^2)$ reflected and transmitted waves propagate in the same direction as their $O(\epsilon)$ counterparts. In other words, to $O(\epsilon^2)$, no deviation from Snell's law or the law of specular reflection is seen. Expressions for the amplitude of the $O(\epsilon^2)$ reflected and transmitted waves have been obtained.

CHAPTER 6

MODIFIED FORMS OF SNELL'S LAW BASED ON SIMPLE WAVE FLOW

6–1 Introduction

Developed in this chapter are two different 'modified forms' of Snell's law—forms which appear to indicate that refraction has a slight amplitude dependence. The 'modified forms' are developed by means other than perturbations. Specifically, the first 'modified form' is obtained by matching the trace velocities of the incident and the transmitted signals at a fixed interface. This 'modified form' was reported by Cotaras and Blackstock (1987). The second 'modified form' is obtained by examining the variation of the pressure along the moving interface and was suggested by Naze Tjøtta and Tjøtta (1988). While different from each other, both forms appear to be correct to second-order. However, in deriving the 'modified forms,' it is assumed that simple wave flow exists in both fluids. We are now, however, in a position to quantify the simple wave flow approximation. Using the results of the previous chapter, we develop the $O(\epsilon)$ and $O(\epsilon^2)$ conditions for simple wave flow to exist in both fluids simultaneously. These conditions are then imposed on 'modified forms.'

This chapter is divided into two sections as follows. In the first section, conditions for simple wave flow in fluids I and II are developed. The $O(\epsilon)$ and $O(\epsilon^2)$ conditions for simple wave flow in fluid I are obtained by determining the conditions for no $O(\epsilon)$ and $O(\epsilon^2)$ reflections. We then examine the conditions required for simple wave flow to exist simultaneously in fluid II. Developed in the econd section of this chapter are the two different 'modified forms' of Snell's law. It turns out that, to $O(\epsilon^2)$, the two methods result in equivalent expressions: ordinary Snell's law and one of the conditions for simple wave flow, which is assumed in their derivation. The 'modified forms' are, therefore, equivalent to ordinary Snell's law.

6–2 Conditions for $O(\epsilon)$ and $O(\epsilon^2)$ Simple Wave Flow

Simple Wave Flow in Fluid I

In this section, the results of the previous chapter are used to determine the $O(\epsilon)$ and $O(\epsilon^2)$ conditions for simple wave flow in fluid I. This is done by determining the $O(\epsilon)$ and $O(\epsilon^2)$ conditions for no reflections. The condition for no $O(\epsilon)$ reflection was noted in the previous chapter [see Eq. (5.70)] and was given the name intromission. Thus, all that remains to be done is to determine the $O(\epsilon^2)$ condition for no reflection. Our procedure is straightforward: First, the $O(\epsilon)$ condition is imposed on the $O(\epsilon^2)$ general solution for fluid I from the previous chapter. The $O(\epsilon^2)$ condition for simple wave flow in fluid I is then determined by forcing the $O(\epsilon^2)$ reflection to vanish.

To impose the $O(\epsilon)$ condition, first recall that the $O(\epsilon^2)$ general solution for fluid I is divided into two parts, a particular solution and a homogeneous solution. At intromission the $O(\epsilon)$ reflection is by definition zero, and, accordingly, the terms in the $O(\epsilon^2)$ particular solution for fluid I that correspond to the interaction of the incident and reflected waves and to the self-action of the reflected wave vanish. The $O(\epsilon^2)$ condition for simple wave flow in fluid I is thus determined solely by forcing the $O(\epsilon^2)$ homogeneous solution for fluid I to vanish.

Imposing the $O(\epsilon)$ condition ($R = 0$) on the terms that make up the $O(\epsilon^2)$ homogeneous solution for fluid I [the terms A through G which are given in Eqs. (5.105)-(5.113)], we obtain the following:

$$A_{\mathrm{reg}} = -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0} \quad , \tag{6.1}$$

$$\begin{aligned} xA_x &= \frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)\frac{x\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0} \\ &= x\frac{\partial A_{\mathrm{reg}}}{\partial x} \quad , \end{aligned} \tag{6.2}$$

$$B = -\frac{1}{2(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\left(1-\frac{n^2}{m}\right)\frac{\partial^2}{\partial t^2}(\Phi_{(1)}^{\mathrm{inc}})^2\Bigg|_{z=0} \quad , \tag{6.3}$$

$$C_{\mathrm{reg}} = 0 \quad , \tag{6.4}$$

$$\begin{aligned} xC_x &= -\frac{\beta^{\mathrm{I}}}{2(c_0^{\mathrm{I}})^2}\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)\frac{x\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{(1)}^{\mathrm{inc}}}{\partial t}\right)^2\Bigg|_{z=0} \\ &= -\frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}x\frac{\partial A_{\mathrm{reg}}}{\partial x} \quad , \end{aligned} \tag{6.5}$$

$$D = \frac{1}{2(c_0^{\mathrm{I}})^2}\left(1-\frac{n^2}{m}\right)\frac{\partial^2}{\partial t^2}(\Phi_{(1)}^{\mathrm{inc}})^2\Big|_{z=0}$$

$$= -\frac{c_0^{\mathrm{I}}}{\cos\theta^{\mathrm{inc}}}B \quad , \tag{6.6}$$

$$E = \frac{1}{(c_0^{\mathrm{I}})^2}\frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\left[\sin^2\theta^{\mathrm{inc}}\left(\frac{m-1}{m}\right)-\left(1-\frac{n^2}{m}\right)\right]\Phi_{(1)}^{\mathrm{inc}}\frac{\partial^2\Phi_{(1)}^{\mathrm{inc}}}{\partial t^2}\Big|_{z=0} \quad , \tag{6.7}$$

$$F = \frac{\cos^2\theta^{\mathrm{inc}}}{(c_0^{\mathrm{I}})^2}(m-1)\Phi_{(1)}^{\mathrm{inc}}\frac{\partial^2\Phi_{(1)}^{\mathrm{inc}}}{\partial t^2}\Big|_{z=0} \quad , \tag{6.8}$$

$$G = \frac{\cos\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\frac{\sin^2\theta^{\mathrm{inc}}}{(c_0^{\mathrm{I}})^2}\left(\frac{m-1}{m}\right)\left(\frac{\partial\Phi^{\mathrm{inc}}}{\partial t}\right)^2\Big|_{z=0} \quad . \tag{6.9}$$

Note that G, the term that accounts for the variation of the normal, is zero for normal incidence. (The condition for intromission at normal incidence is $m = n$.) For all other angles, however, m must be unity for G to be zero. However, if m is unity, then the $O(\epsilon)$ condition [see Eq. (5.70)] in combination with Snell's law indicates that n is unity also, that is, $n = m = 1$. If both m and n are unity, then E, F, and G, which together account for all interface motion, are zero. Note that even with n and m equal to unity, the term A_{reg} is not zero unless $\beta^{\mathrm{I}} = \beta^{\mathrm{II}}$.

We now determine the form of the $O(\epsilon^2)$ homogeneous solution given that $R = 0$. Using Eqs. (6.2) and (6.5) in Eqs. (5.148) and (5.149), we see that for the special case of $R = 0$

$$\Phi_{(2)a}^{\mathrm{refl}}\Big|_{z=0} = 0 \quad , \tag{6.10}$$

$$\Phi_{(2)a}^{\mathrm{trans}}\Big|_{z=0} = \frac{n}{m^2}\frac{\sin\theta^{\mathrm{inc}}}{\cos^2\theta^{\mathrm{inc}}}A_{\mathrm{reg}} \quad . \tag{6.11}$$

Note that $\Phi_{(2)a}^{\mathrm{trans}}$, the part of the homogeneous solution for fluid II that has amplitude variation along its wavefront, is zero in the case of normal incidence. For oblique incidence, $\Phi_{(2)a}^{\mathrm{trans}}$ is zero only if A_{reg} is zero. Use of Eqs. (6.6), (6.10), and (6.11) in Eqs. (5.154) and (5.155) leads to

$$\frac{\partial\Phi_{(2)b}^{\mathrm{refl}}}{\partial t}\Big|_{z=0} = -\frac{c_0^{\mathrm{I}}}{2\cos\theta^{\mathrm{inc}}}\left[\left(1+\frac{\tan^2\theta^{\mathrm{inc}}}{m^2}\right)A_{\mathrm{reg}}+E+G\right]+\frac{F}{2} \tag{6.12}$$

and

$$\frac{\partial\Phi_{(2)b}^{\mathrm{trans}}}{\partial t}\Big|_{z=0} = \frac{T}{m}\frac{\partial\Phi_{(2)}^{\mathrm{inc}}}{\partial t}-\frac{c_0^{\mathrm{I}}}{2\cos\theta^{\mathrm{inc}}}\frac{1}{m}\left(1+\frac{\tan^2\theta^{\mathrm{inc}}}{m^2}\right)A_{\mathrm{reg}}$$

$$-\frac{c_0^{\mathrm{I}}}{2\cos\theta^{\mathrm{inc}}}\frac{1}{m}(2B+E+G)-\frac{F}{2m} \quad . \tag{6.13}$$

Equations (6.10) and (6.12) are the two terms that make up the $O(\epsilon^2)$ homogeneous solution for fluid I. Since Eq. (6.10) shows that $\Phi^{\mathrm{refl}}_{(2)a}$ is already zero, the $O(\epsilon^2)$ conditions for simple wave flow in fluid I may be obtained by forcing $\Phi^{\mathrm{refl}}_{(2)b}$ to be zero.

An examination of Eqs. (6.1), (6.7), (6.8), and (6.9) indicates that for $\Phi^{\mathrm{refl}}_{(2)b}$ to be zero, the coefficients of $(\partial\Phi^{\mathrm{inc}}/\partial t)^2$ and $\Phi^{\mathrm{inc}}_{(1)}\partial^2\Phi^{\mathrm{inc}}_{(1)}/\partial t^2$ must be zero. Setting the coefficient of $\Phi^{\mathrm{inc}}_{(1)}\partial^2\Phi^{\mathrm{inc}}_{(1)}/\partial t^2$ equal to zero yields the $O(\epsilon)$ condition for intromission, which was previously assumed. No new conditions are therefore required to force the coefficient of $\Phi^{\mathrm{inc}}_{(1)}\partial^2\Phi^{\mathrm{inc}}_{(1)}/\partial t^2$ to be zero. On the other hand, forcing the coefficient of $(\partial\Phi^{\mathrm{inc}}/\partial t)^2$ to be zero leads to the following condition:

$$\frac{\beta^{\mathrm{I}}}{2}\left(1+\frac{\tan^2\theta^{\mathrm{inc}}}{m^2}\right)\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)+\sin^2\theta^{\mathrm{inc}}\left(\frac{1-m}{m}\right)=0 \quad . \tag{6.14}$$

Note that at normal incidence, Eq. (6.14) reduces to (with $m=n$)

$$\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)=0 \quad . \tag{6.15}$$

Equation (6.15) indicates that for no $O(\epsilon^2)$ reflection at normal incidence, the jump in the quantity β/c_0 must be zero. For oblique incidence the condition is, however, more complicated. The angle dependence in Eq (6.14) may be removed using the $O(\epsilon)$ condition [see Eq. (5.71)], and we thereby obtain the following form of the condition for no $O(\epsilon^2)$ reflection:

$$\frac{\beta^{\mathrm{I}}}{2}\left(1-\frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right)+\frac{m(n^2-1)(n^2-m^2)}{n^2(m+1)(m^2-1)}=0 \quad . \tag{6.16}$$

Thus, for oblique incidence, simple wave flow exists in fluid I if the properties of the fluid pair are such that both the $O(\epsilon)$ and $O(\epsilon^2)$ conditions for no reflections [Eqs. (5.71) and (6.16), respectively] are met simultaneously. For normal incidence, the number of conditions for simple wave flow is the same—two, but the conditions themselves are simplier in form. The $O(\epsilon)$ condition for no reflection at normal incidence is obtained from Eq. (5.70),

$$[\![\rho_0 c_0]\!]=0 \quad . \tag{6.17}$$

The $O(\epsilon^2)$ condition for no reflection at normal incidence is Eq. (6.15).

Simple Wave Flow in Fluids I and II

We now examine the conditions for simple wave flow to exist simultaneously in fluids I and II. We start by considering what restriction the simple wave flow assumption places on the relationship between the temporal and spatial derivatives of an acoustic field variable. The restriction is central to many relations that are derived assuming simple wave flow.

As an example, we develop the nonlinear impedance relation for the special case of outgoing simple waves. Our starting point is the second-order form of the momentum equation, Eq. (2.88). If simple wave flow in a lossless fluid is assumed, Eq. (2.88) simplifies to

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = 0 \quad . \tag{6.18}$$

If the acoustic field consists of simple outgoing waves, the time and space derivatives of the pressure are related by

$$\nabla p' = -\frac{\mathbf{n}}{c_0 + \beta(\mathbf{n} \cdot \mathbf{u})} \frac{\partial p'}{\partial t} \quad , \tag{6.19}$$

where $\mathbf{n}$ is the direction of propagation. [Recall that the propagation speed of a finite-amplitude wave is $c_0 + \beta(\mathbf{n} \cdot \mathbf{u})$.] Substituting the foregoing in Eq. (6.18), integrating, and noting that the integration constant must be zero in order to satisfy quiet conditions leads to

$$p' = \rho_0 \left(c_0 + \frac{\beta(\mathbf{n} \cdot \mathbf{u})}{2} \right) (\mathbf{n} \cdot \mathbf{u}) \quad . \tag{6.20}$$

Equation (6.20) is the second-order form of the impedance relation and is valid for simple outgoing waves of finite amplitude.

Equation (6.19) indicates that, under the simple wave flow assumption, the temporal and spatial derivatives are *simply* related; moreover, the gradient of the pressure is parallel to the direction of propagation. The particle motion is, accordingly, restricted to being parallel to the direction of propagation.

To obtain a condition for simple wave flow in fluid II, we examine the gradient of the $O(\epsilon)$ and $O(\epsilon^2)$ pressures in fluid II and demand that they be parallel to the direction of propagation. The gradient of the $O(\epsilon)$ pressure in fluid II is obtained by taking the gradient of Eq. (5.27). Noting Eq. (5.50), we find that, to $O(\epsilon)$, the gradient of the pressure is parallel to the direction of propagation. Thus no additional conditions are required to assume simple wave

flow exists in fluid II to $O(\epsilon)$. The situation is more complicated at $O(\epsilon^2)$. The $O(\epsilon^2)$ pressure in fluid II is given by Eq. (5.28). The second term on the right-hand side of Eq. (5.28) is a quadratic form of the $O(\epsilon)$ solution, and the gradient of this term is, accordingly, parallel to the direction of propagation. The first term on the right-hand side of Eq. (5.28) may be divided into two parts: a particular solution and a homogeneous solution. From Eq. (5.86) we see that the fluid II particular solution is also a quadratic form of the $O(\epsilon)$ solution, the gradient of which is parallel to the direction of propagation. The fluid II homogeneous solution is given by Eq. (5.118). By inspection we see that for the gradient of $\Phi^{\mathrm{II}}_{(2)h}$ to be parallel to the direction of propagation, $\Phi^{\mathrm{trans}}_{(2)a}$ must be zero. We noted earlier that, for the special case of $R = 0$, $\Phi^{\mathrm{trans}}_{(2)a}$ is zero for normal incidence, but for oblique incidence, $\Phi^{\mathrm{trans}}_{(2)a}$ is zero only if A_{reg} is zero. Furthermore, A_{reg} is zero only if

$$\left(1 - \frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right) = 0 \quad . \tag{6.21}$$

Since Eq. (6.21) is the same as the normal incidence condition for $O(\epsilon^2)$ simple wave flow in fluid I [Eq. (6.15)], no additional conditions are required for $O(\epsilon^2)$ simple wave flow to exist simultaneously in fluids I and II at normal incidence. For oblique incidence, however, Eq. (6.21) represents an additional condition. Thus, for $O(\epsilon^2)$ simple wave flow to exist simultaneously in both fluids I and II at oblique incidence, the conditions expressed in Eqs. (5.71), (6.14), and (6.21) must be met simultaneously. The only way all three conditions may be satisfied simultaneously is if the static densities, the small-signal sound speeds, and the coefficients of nonlinearity are matched,

$$\rho_0^{\mathrm{I}} = \rho_0^{\mathrm{II}} \quad , \tag{6.22}$$

$$c_0^{\mathrm{I}} = c_0^{\mathrm{II}} \quad , \tag{6.23}$$

$$\beta^{\mathrm{I}} = \beta^{\mathrm{II}} \quad . \tag{6.24}$$

6–3 Modified Forms of Snell's Law

In this section, two 'modified forms' of Snell's law are developed. The first is obtained by matching the trace velocities of the incident and transmitted signals at the interface, while assuming that the interface is stationary. The second 'modified form' of Snell's law is obtained by examining the variation of the pressure along the moving interface. The $O(\epsilon)$ and $O(\epsilon^2)$ results of the two methods are found to be equivalent. Moreover, when the conditions for simple

wave flow, which is assumed in their derivation, are imposed on the 'modified forms', they reduce to ordinary Snell's law.

Trace velocity matching[1]

This section progresses as follows: First, ordinary Snell's law is derived by matching the trace velocities of small-signal waves. Then, a 'modified form' of Snell's law is developed by matching the trace velocities of finite amplitude waves at the interface. Simple wave flow is assumed on the fluid I side, and the interface is assumed stationary. The $O(\epsilon)$ and $O(\epsilon^2)$ approximations of the 'modified form' are then developed.

The method of matching the trace velocities of the incident and transmitted signals is first demonstrated by deriving Snell's law for small-signal waves. Shown in Fig. 6.1 is a plane wave obliquely incident on a plane interface. First, consider fluid I. We follow the progress of a single typical wavelet of the incident wave, in this case, a peak. At time $t = 0$, the wavefront for this wavelet is AA'. At a time Δt later, the wavefront has moved a distance $c_0^{\mathrm{I}}\Delta t$ to position BB'. The increment the wavelet has moved along the interface, that is, the trace distance Δx_{AB} is

$$\Delta x_{AB} = \frac{c_0^{\mathrm{I}}\Delta t}{\sin\theta^{\mathrm{inc}}} \quad . \tag{6.25}$$

Note that $c_0^{\mathrm{I}}/\sin\theta^{\mathrm{inc}}$ is the trace velocity of the wavelet along the interface. Next consider fluid II. The wavelet of the transmitted wave at time $t = 0$ is represented by the wavefront AA'' and, at a time Δt later, by BB''. The distance along the ray is $c_0^{\mathrm{II}}\Delta t$ and the trace distance is given by

$$\Delta x_{AB} = \frac{c_0^{\mathrm{II}}\Delta t}{\sin\theta^{\mathrm{trans}}} \quad . \tag{6.26}$$

Thus, the trace velocity is $c_0^{\mathrm{II}}/\sin\theta^{\mathrm{trans}}$. Combining the two equations leads to Snell's law,

$$\frac{c_0^{\mathrm{I}}}{\sin\theta^{\mathrm{inc}}} = \frac{c_0^{\mathrm{II}}}{\sin\theta^{\mathrm{trans}}} \quad . \tag{6.27}$$

It appears that Snell's law is simply a kinematic condition that requires the trace velocities to be the same for both the incident and transmitted waves. (Note

[1] The trace velocity matching technique and the resulting 'modified form' of Snell's law were presented at the 114th meeting of the Acoustical Society of America (Cotaras and Blackstock 1987).

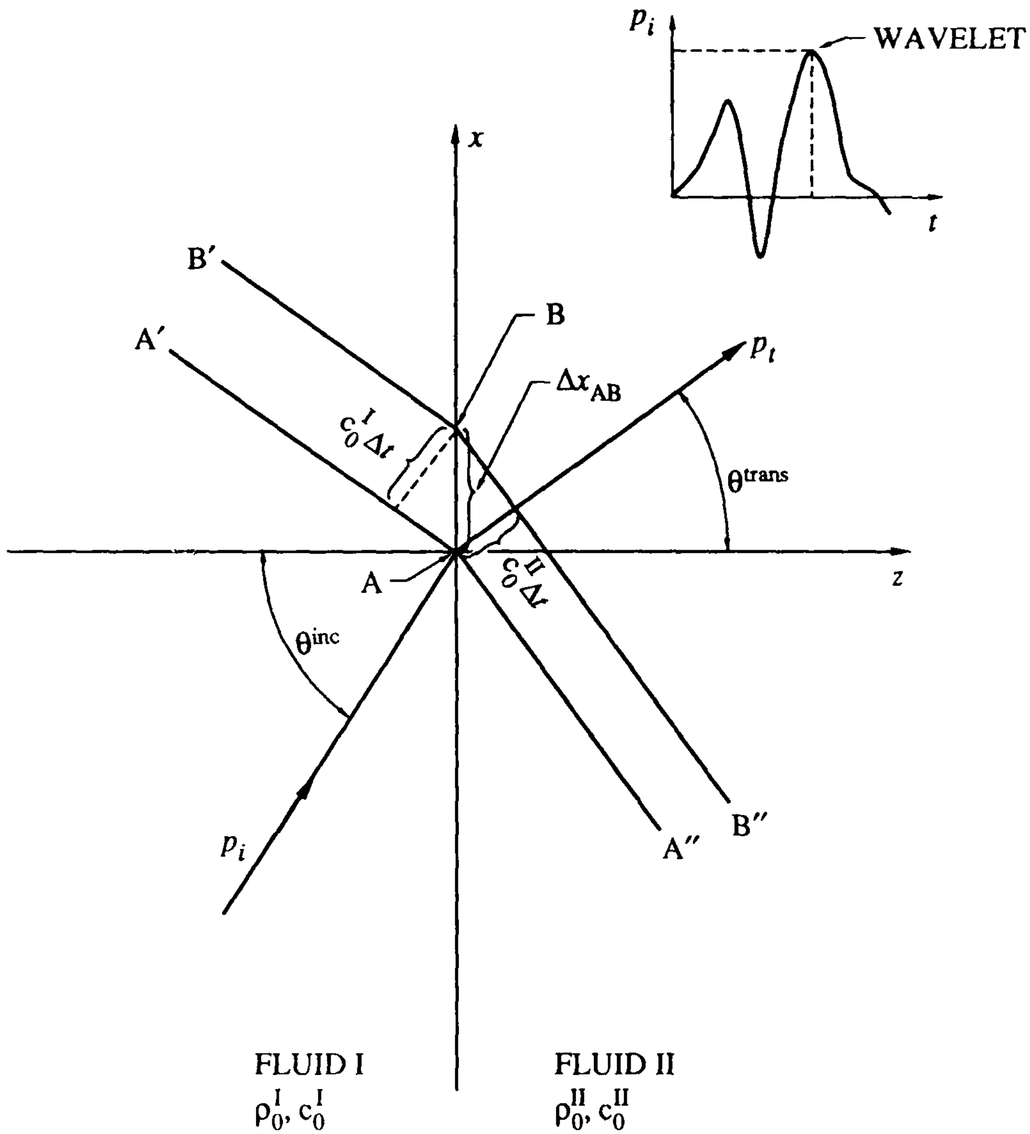

Figure 6.1 SMALL-SIGNAL PLANE WAVE OBLIQUELY INCIDENT ON A PLANE FLUID-FLUID INTERFACE

that the trace velocity matching method may also be used to obtain the law of specular reflection.)

The trace velocity matching technique is now used for finite-amplitude waves. Recall that the propagation velocity of a simple outgoing wave of finite amplitude is $c_0^{\mathrm{I}} + \beta^{\mathrm{I}}(\mathbf{n} \cdot \mathbf{u})$—not just the small-signal sound speed c_0^{I}. Consider Fig. 6.2. By invoking the same geometrical argument used for small-signal waves, we obtain the following 'modified form' of Snell's law for finite amplitude waves,

$$\frac{\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}} + \beta^{\mathrm{I}}(\mathbf{n}^{\mathrm{inc}} \cdot \mathbf{u}^{\mathrm{inc}})} = \frac{\sin\theta^{\mathbf{trans}}}{c_0^{\mathrm{II}} + \beta^{\mathrm{II}}(\mathbf{n}^{\mathbf{trans}} \cdot \mathbf{u}^{\mathbf{trans}})} \quad . \tag{6.28}$$

Note, however, that simple wave motion was assumed in fluid I and that the motion of the interface was neglected.

The $O(\epsilon)$ and $O(\epsilon^2)$ forms of the 'modified form' of Snell's law are now developed. We first expand Eq. (6.28) using the techniques developed in Chapter 5, specifically, Eqs. (5.11)–(5.13). Using Eqs. (5.12) and (5.13) in Eq. (6.28), then binomially expanding the denominators and retaining only the leading and next higher-order terms leads to

$$\frac{\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}}\left(1 - \epsilon\,\frac{\beta^{\mathrm{I}}}{c_0^{\mathrm{I}}}(\mathbf{n}^{\mathrm{inc}} \cdot \mathbf{u}_{(1)}^{\mathrm{inc}})\right) = \frac{\sin\theta^{\mathbf{trans}}}{c_0^{\mathrm{II}}}\left(1 - \epsilon\,\frac{\beta^{\mathrm{II}}}{c_0^{\mathrm{II}}}(\mathbf{n}^{\mathbf{trans}} \cdot \mathbf{u}_{(1)}^{\mathbf{trans}})\right) \quad . \tag{6.29}$$

Equating the leading-order terms in Eq. (6.29) yields the following $O(\epsilon)$ form of the 'modified form' of Snell's law:

$$\frac{\sin\theta^{\mathrm{inc}}}{c_0^{\mathrm{I}}} = \frac{\sin\theta^{\mathbf{trans}}}{c_0^{\mathrm{II}}} \quad .$$

This is merely the ordinary form of Snell's law. It was noted in the previous chapter that Snell's law is one of the conditions required for the $O(\epsilon)$ solutions for the wave equation to be valid for all x and all t. Equating the next higher-order terms of Eq. (6.29) and using Snell's law leads to the $O(\epsilon^2)$ form of the 'modified form' of Snell's law,

$$\left(1 - \frac{\beta^{\mathrm{II}}}{\beta^{\mathrm{I}}}\frac{n^2}{m}\right) = 0 \quad . \tag{6.30}$$

In obtaining Eq. (6.30), we have made use of the following $O(\epsilon)$ result, which was in turn obtained using the $O(\epsilon)$ pressure balance, Eq. (5.35), and the $O(\epsilon)$ impedance relation, $p = \rho_0 c_0(\mathbf{n} \cdot \mathbf{u})$:

$$\mathbf{n}^{\mathrm{inc}} \cdot \mathbf{u}_{(1)}^{\mathrm{inc}} = \frac{n}{m}\,\mathbf{n}^{\mathrm{trans}} \cdot \mathbf{u}_{(1)}^{\mathrm{trans}} \quad . \tag{6.31}$$

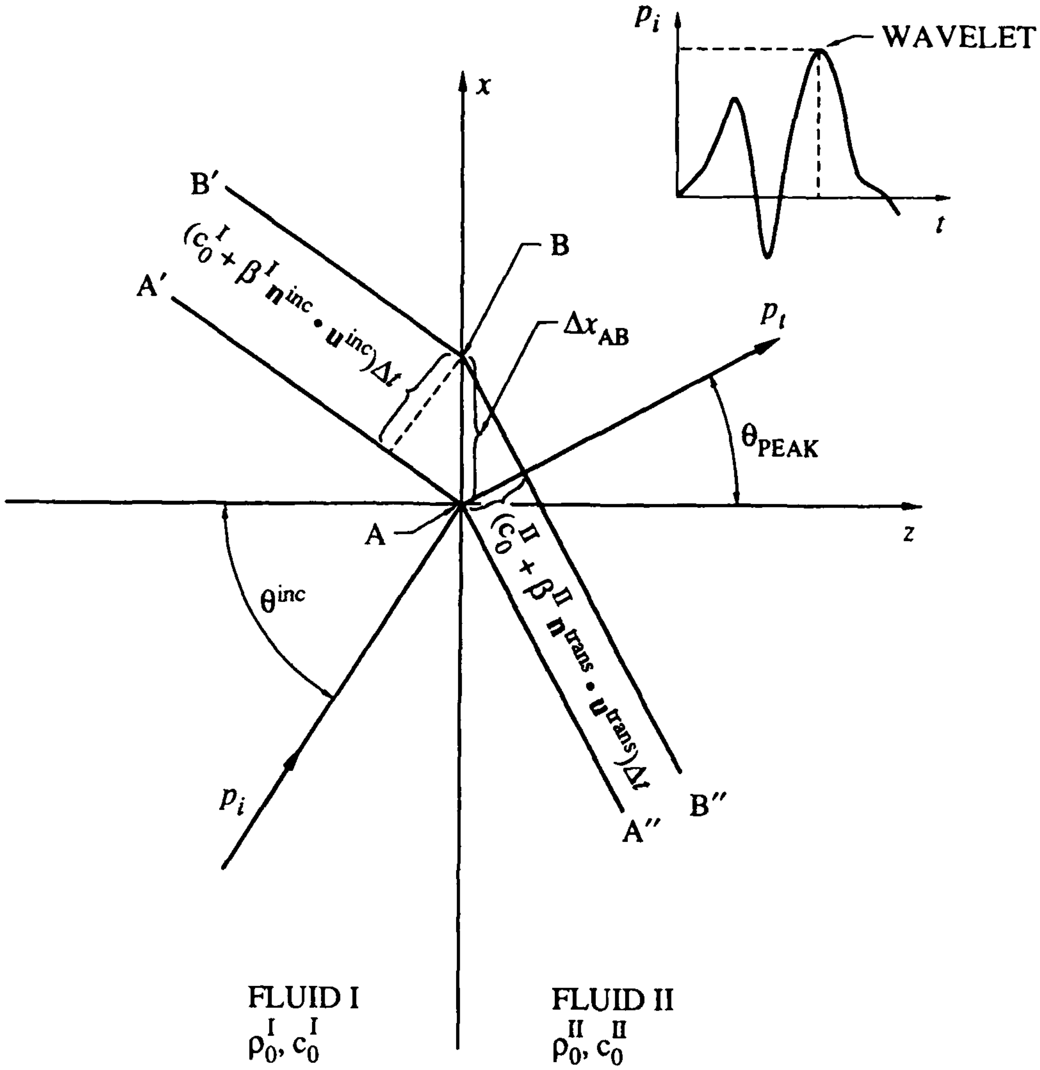

Figure 6.2 FINITE-AMPLITUDE PLANE WAVE OBLIQUELY INCIDENT ON A PLANE FLUID-FLUID INTERFACE

It was noted in the previous section, however, that Eq. (6.30) is a condition for simple wave flow to exist in fluid II to $O(\epsilon^2)$; see Eq. (6.21). Thus, to $O(\epsilon^2)$, the 'modified form' of Snell's law and the ordinary form are equivalent. That is to say, the 'modified form' of Snell's law is a trivial *non-extension* of Snell's law. The problem with the derivation of the 'modified form' is that the conditions imposed by the simple wave flow assumption were overlooked.

In summary, it has been shown that, to $O(\epsilon^2)$, the 'modified form' of Snell's law that is obtained by matching the trace velocities reduces to ordinary Snell's law when the conditions of the simple wave flow are imposed on it.

Variation of pressure along the interface[2]

In this section, another 'modified form' of Snell's law is derived—this one by examining the variation of the pressure along the interface. A Taylor series expansion is then used to obtain $O(\epsilon)$ and $O(\epsilon^2)$ approximation of this 'modified form' along $z = 0$. Simple wave flow is, however, again assumed on the fluid I side, and it turns out that this 'modified form' of Snell's law yields the same $O(\epsilon)$ and $O(\epsilon^2)$ expressions that were obtained using the trace velocity method.

We first show that the jump in the variation of the pressure along the interface is zero. A simple way to show this is to describe the interface in a parametric form using only a length coordinate ℓ and the time t. The pressure jump at a point ℓ is then written

$$p^{\mathrm{I}}(\ell,t) - p^{\mathrm{II}}(\ell,t) = 0 \quad . \tag{6.32}$$

Similarly, at a point $\ell + \Delta\ell$, we may write

$$p^{\mathrm{I}}(\ell+\Delta\ell,t) - p^{\mathrm{II}}(\ell+\Delta\ell,t) = 0 \quad . \tag{6.33}$$

Combining the two previous expressions and dividing by $\Delta\ell$ yields the desired result,

$$\frac{\partial p^{\mathrm{I}}}{\partial \ell} = \frac{\partial p^{\mathrm{II}}}{\partial \ell} \quad . \tag{6.34}$$

The variation of the pressure along the interface may also be expressed in terms of the tangent to the interface $\mathbf{T}$ and the gradient of the pressure,

$$\mathbf{T}\cdot\boldsymbol{\nabla} p^{\mathrm{I}} = \mathbf{T}\cdot\boldsymbol{\nabla} p^{\mathrm{II}} \quad . \tag{6.35}$$

The second 'modified form' of Snell's law may be obtained by applying Eq. (6.35) along the moving interface. If simple wave flow is assumed on the

[2]This method was suggested by Naze Tjøtta and Tjøtta (1988).

fluid I side, Eq. (6.19) may be used to obtain an expression for the gradient. Thus, along the interface, Eq. (6.35) may be written as

$$\frac{\sin\theta^{\text{inc}}}{c_0^{\text{I}}+\beta^{\text{I}}(\mathbf{n}^{\text{inc}}\cdot\mathbf{u}^{\text{inc}})}\frac{\partial p^{\text{inc}}}{\partial t}=\frac{\sin\theta^{\text{trans}}}{c_0^{\text{II}}+\beta^{\text{II}}(\mathbf{n}^{\text{trans}}\cdot\mathbf{u}^{\text{trans}})}\frac{\partial p^{\text{trans}}}{\partial t}\quad, \tag{6.36}$$

where the angles θ^{inc} and θ^{trans} are defined relative to the instantaneous normal. Except for the $\partial p/\partial t$ terms, Eq. (6.36) is the same as our previous 'modified form' of Snell's law, Eq. (6.28), which was obtained neglecting interface motion altogether.

If this 'modified form' of Snell's law is to be compared with the previous one, the angles must be referred to the $+z$-axis and the displacement of the interface must be accounted for. Moreover, the entire expression must be approximated to $O(\epsilon^2)$. This may be accomplished as follows: First, an expression for the tangent is obtained using the previously derived $O(\epsilon)$ expression for the normal to the interface, Eq. (4.19). Next, an $O(\epsilon^2)$ expression for the gradient of the pressure is obtained by expanding Eq. (6.19) and using a Taylor series expansion about $z=0$. This, of course, assumes simple wave flow on the fluid I side. The two expressions are then combined to yield the $O(\epsilon)$ and $O(\epsilon^2)$ forms of Eq. (6.35) about $z=0$.

An expression for the tangent may be obtained from our expression for the normal to the interface $\mathbf{N}$, Eq. (4.19). Since $\mathbf{T}$ must be perpendicular to $\mathbf{N}$, $\mathbf{T}$ is given by

$$\mathbf{T}=\pm\,\mathbf{i}\pm\mathbf{k}\,\epsilon\int\frac{\partial w_{(1)}^{\text{trans}}}{\partial x}\,dt\quad. \tag{6.37}$$

The $+$ sign is chosen.

If simple wave flow is assumed on the fluid I side, Eq. (6.19) may be used to obtain an expression for the gradient. Expanding the denominator binominally and accounting for the motion of the interface using a Taylor series expansion results in the following expression for the gradient of the pressure along $z=0$:

$$\begin{aligned}\nabla p^{\text{I}} &= -\frac{\mathbf{n}^{\text{inc}}}{c_0^{\text{I}}}\left\{\epsilon\left.\frac{\partial p_{(1)}^{\text{inc}}}{\partial t}\right|_{z=0}\right.\\ &\left.+\,\epsilon^2\left(\frac{\partial p_{(2)}^{\text{inc}}}{\partial t}+\frac{\partial^2 p_{(1)}^{\text{inc}}}{\partial t\,\partial z}\int w_{(1)}^{\text{trans}}\,dt-\frac{\beta^{\text{I}}}{c_0^{\text{I}}}\frac{\partial p_{(1)}^{\text{inc}}}{\partial t}(\mathbf{n}^{\text{inc}}\cdot\mathbf{u}^{\text{inc}})\right)\right|_{z=0}\right\}\ . \end{aligned}\tag{6.38}$$

The expression for the gradient of the pressure in fluid II is similar.

Using Eqs. (6.37) and (6.38) in Eq. (6.35) and equating equal powers of ϵ yields the following $O(\epsilon)$ and $O(\epsilon^2)$ results:

$$\frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\left.\frac{\partial p_{(1)}^{\rm inc}}{\partial t}\right|_{z=0} = \frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\left.\frac{\partial p_{(1)}^{\rm trans}}{\partial t}\right|_{z=0} \quad , \tag{6.39}$$

$$\begin{aligned}
&-\frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\frac{\partial p_{(2)}^{\rm trans}}{\partial t} - \frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\frac{\partial^2 p_{(1)}^{\rm inc}}{\partial t\partial z}\int w_{(1)}^{\rm trans}\,dt \\
&\quad -\frac{\cos\theta^{\rm inc}}{c_0^{\rm I}}\frac{\partial p_{(1)}^{\rm inc}}{\partial t}\int\frac{\partial w_{(1)}^{\rm trans}}{\partial x}\,dt + \frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\frac{\beta^{\rm I}}{c_0^{\rm I}}\frac{\partial p_{(1)}^{\rm inc}}{\partial t}(\mathbf{n}^{\rm inc}\cdot\mathbf{u}_{(1)}^{\rm inc}) \\
&= -\frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\frac{\partial p_{(2)}^{\rm trans}}{\partial t} - \frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\frac{\partial^2 p_{(1)}^{\rm trans}}{\partial t\,\partial z}\int w_{(1)}^{\rm trans}\,dt \\
&\quad -\frac{\cos\theta^{\rm trans}}{c_0^{\rm II}}\frac{\partial p_{(1)}^{\rm trans}}{\partial t}\int\frac{\partial w_{(1)}^{\rm trans}}{\partial x}\,dt + \frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\frac{\beta^{\rm II}}{c_0^{\rm II}}\frac{\partial p_{(1)}^{\rm trans}}{\partial t}(\mathbf{n}^{\rm trans}\cdot\mathbf{u}_{(1)}^{\rm trans}) \quad .
\end{aligned} \tag{6.40}$$

Fortunately, the $O(\epsilon)$ and $O(\epsilon^2)$ results given in Eqs. (6.39) and (6.40) may be simplified. Use of the time derivative of the $O(\epsilon)$ pressure balance given in Eq. (5.35) reduces Eq. (6.40) to Snell's law,

$$\frac{\sin\theta^{\rm inc}}{c_0^{\rm I}} = \frac{\sin\theta^{\rm trans}}{c_0^{\rm II}} \quad .$$

Equation (6.40) may be simplified by noting that, for simple wave flow, the x-derivative of the $O(\epsilon^2)$ pressure balance, Eq. (5.36), is

$$\begin{aligned}
&-\frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\left.\frac{\partial p_{(2)}^{\rm inc}}{\partial t}\right|_{z=0} - \frac{\sin\theta^{\rm inc}}{c_0^{\rm I}}\frac{\partial^2 p_{(1)}^{\rm inc}}{\partial t\,\partial z}\left.\int w_{(1)}^{\rm trans}\,dt\right|_{z=0} \\
&\quad -\frac{\cos\theta^{\rm inc}}{c_0^{\rm I}}\frac{\partial p_{(1)}^{\rm inc}}{\partial t}\left.\int\frac{\partial w_{(1)}^{\rm trans}}{\partial x}\,dt\right|_{z=0} \\
&= -\frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\left.\frac{\partial p_{(2)}^{\rm trans}}{\partial t}\right|_{z=0} - \frac{\sin\theta^{\rm trans}}{c_0^{\rm II}}\frac{\partial^2 p_{(1)}^{\rm trans}}{\partial t\,\partial z}\left.\int w_{(1)}^{\rm trans}\,dt\right|_{z=0} \\
&\quad -\frac{\cos\theta^{\rm trans}}{c_0^{\rm II}}\frac{\partial p_{(1)}^{\rm trans}}{\partial t}\left.\int\frac{\partial w_{(1)}^{\rm trans}}{\partial x}\,dt\right|_{z=0} \quad .
\end{aligned} \tag{6.41}$$

Use of Eqs. (6.41), (6.31), and Snell's law reduces Eq. (6.40) to

$$\left(1 - \frac{\beta^{\rm II}}{\beta^{\rm I}}\frac{n^2}{m}\right) = 0 \quad .$$

This is the same expression obtained using the trace velocity approach, Eq. (6.30).

Thus, the 'modified form' of Snell's law that is obtained by examining the variation of the pressure along the interface is, to $O(\epsilon^2)$, equivalent to that obtained by matching the trace velocities if the results are referred to $z = 0$. Moreover, both 'modified forms' are, to $O(\epsilon^2)$, equivalent to ordinary Snell's law. This implies that neglecting the motion of the boundary in the derivation of the first 'modified form' is not a problem. The problem is the simple wave flow assumption that is made when deriving both of the modified forms.

6–4 Summary

In this chapter, we examined the conditions for $O(\epsilon)$ and $O(\epsilon^2)$ simple wave flow in fluid I and fluid II. For oblique incidence, the conditions are that the static densities, the small-signal sound speeds, and the coefficients of nonlinearity be matched. For normal incidence it is only required that $\rho_0 c_0$ and $\rho_0 c_0^2/\beta$ be matched. Also examined in this chapter are two 'modified forms' of Snell's law. It was found that the two 'modified forms' were, to $O(\epsilon^2)$, equivalent and that they reduce to ordinary Snell's law and one of the conditions for simple wave flow, which was assumed in their derivation. Thus, to $O(\epsilon^2)$, they are equivalent to ordinary Snell's law.

CHAPTER 7

SUMMARY OF RESULTS AND PROPOSALS FOR FUTURE WORK

7–1 Summary

This dissertation is divided into three parts: In the first part the basic equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism were examined. The examination led to the following results: (1) Simplified (correct to second order) forms of the equations for a homogeneous, thermoviscous fluid with a single relaxation mechanism were developed. (2) The wave equation for finite-amplitude wave motion in such a fluid was also developed. The derivation of this wave equation from first principles has not been presented previously. The following assumptions were made during the derivation: (1) The deviation from equilibrium was assumed to be small, and, accordingly, linear relations between the thermodynamic fluxes and forces were used. (2) The possibility of a cross-effect between bulk viscosity and relaxation was neglected. (3) The magnitude of the acoustic signal was assumed to be small, but large enough to make quadratic nonlinearity terms significant. (4) The magnitude of transport coefficients was assumed to be small, and (5) the amount of dispersion was also assumed to be small.

In the second part of the dissertation, the reflection and refraction of the finite-amplitude sound at an interface between two lossless fluids was analyzed. The analysis yielded the following results: (1) Expressions that are correct to second order were derived for the kinematic and force balance boundary conditions. (2) Expressions for the $O(\epsilon)$ and $O(\epsilon^2)$ reflected and transmitted fields were obtained, and specific terms that account for the displacement of the interface, the variation of the normal to the interface, and the $O(\epsilon^2)$ source condition were identified. (3) It was noted that, to $O(\epsilon^2)$, no deviation from Snell's law or the law of specular reflection is predicted. None of these expressions have

been previously obtained, and all expressions were obtained for an arbitrary (not necessarily harmonic) source condition.

The significant results of the third part of this dissertation are as follows: (1) The requirements for simple wave flow to exist simultaneously in fluids I and II are that the small-signal sound speeds, the static densities, and the coefficients of nonlinearity be matched. (2) The two 'modified forms' of Snell's law are, to $O(\epsilon^2)$, equivalent. (3) When the conditions of simple wave flow in fluids I and II are imposed on the 'modified forms' of Snell's law, they reduce to ordinary Snell's law.

7–2 Future Work

The work presented in this dissertation is much less extensive than the work that remains to be done. Two ideas for future work on the fundamental assumptions in the derivation of the basic equations are the following: The first is to examine the possibility of nonlinear thermodynamic effects. Such effects occur because the deviation from equilibrium is not small enough to be correctly predicted by a linear model. For the case of a chemical reaction as the relaxation mechanism, Prigogine (1961) points out that the deviation from equilibrium is frequently large enough to require the use of nonlinear relations between the thermodynamic fluxes and forces. The second idea for future work is to include the effects of diffusion. Diffusion may be important near the interface of two fluids. Moreover, the effects of diffusion are known to interact with the effects of heat conduction; see Prigogine (1961).

Future work on the problem of reflection and refraction of finite-amplitude sound should be oriented towards reducing the idealizations of this work by performing the two following extensions. (1) The effects of losses in the fluids should be included in both the wave equations and the boundary conditions. (2) The effects of finite source size should be included by using results of this dissertation as one of the components in the spatial Fourier decomposition of the source. The results of this work may, however, be readily used to analyze two special cases that are easily realized experimentally, namely, reflections from a rigid wall ($R = 1$) and reflections from a pressure release surface ($R = -1$). In the long term, a numerical implementation of the results would be useful for computing the transmitted and reflected fields of a given source, both near and far from the interface. In this way many different source-receiver geometries and fluid pairs could be examined.

APPENDIX A

THERMODYNAMICS OF RELAXING FLUIDS

A–1 Introduction

It is useful to know the origins of the fundamental thermodynamic relations that apply to acoustics. The majority of the expressions that are developed in this appendix are well known for nonrelaxing fluids (see, for example, Van-Wylen and Sonntag (1976)), but not so well known for relaxing fluids. Good references on the thermodynamics of relaxing fluids and its application to acoustics do, however, exist; see, for example, de Groot and Mazur (1962) and Bhatia (1967). The interested reader is referred to these references for more details. The purpose of deriving fundamental relations in this appendix is merely completeness.

The notation used in this appendix is as follows: Frozen quantities are denoted by a superscript $^{\infty}$, equilibrium values by a superscript ^{0}. The reason for this, which is elaborated in the main text, is that the frozen value is the correct value for infinite frequency; the equilibrium values are correct for zero frequency. The notation used to denote the static value of any thermodynamic variable is the subscript $_0$. If a fluid does not contain a relaxation mechanism, no difference between the frozen and equilibrium states exists. The notation used to denote variables in this situation is the same, except the superscripts are dropped. For example, the static values of the specific heats at constant pressure and constant volume for a nonrelaxing fluid are denoted c_{p_0} and c_{v_0}, respectively. All variables in this appendix are dimensional, and, consequently, no special notation is required to indicate dimensionality.

A–2 Relations from Gibbs Equation

Considered in this section are the relations that may be obtained from Gibbs equation. Use of Gibbs equation leads to differential equations for the

pressure, density, and temperature. Relations between derivatives of the temperature and pressure may then be obtained. We start by stating one form of Gibbs equation,

$$de = Tds + \frac{P}{\rho^2}\,d\rho - A\,dq \quad , \tag{A.1}$$

where, as in the main text, e is the specific internal energy (energy per unit mass), T is the absolute temperature, s is the entropy (per unit mass), P is the total pressure, ρ is the density, A is the affinity of the relaxation process, and q is the degree of advancement of the relaxation mechanism. As was pointed out in the main text, three independent state variables are required for relaxing fluids. In general, ρ, s, and q are used as the independent variables in this work. Accordingly, the differential of the internal energy e may be expressed as follows:

$$de = \left.\frac{\partial e}{\partial s}\right|_{\rho,q} ds + \left.\frac{\partial e}{\partial \rho}\right|_{s,q} d\rho + \left.\frac{\partial e}{\partial q}\right|_{\rho,s} dq \quad . \tag{A.2}$$

Comparison with Eq. (A.1) reveals that

$$T = \left.\frac{\partial e}{\partial s}\right|_{\rho,q} \quad , \tag{A.3}$$

$$P = \rho^2 \left.\frac{\partial e}{\partial \rho}\right|_{s,q} \quad , \tag{A.4}$$

and

$$A = -\left.\frac{\partial e}{\partial q}\right|_{\rho,s} \quad . \tag{A.5}$$

Cross-differentiation between Eqs. (A.3) and (A.4) reveals that

$$\left.\frac{\partial T}{\partial \rho}\right|_{s,q} = \frac{1}{\rho^2}\left.\frac{\partial P}{\partial s}\right|_{\rho,q} \quad . \tag{A.6}$$

Equations (A.3), (A.4), and (A.6) are valid for constant, or frozen, q.

Equations (A.1) and (A.2) are now examined at equilibrium. At equilibrium, q takes on its equilibrium value q^*, and A is equal to zero. Furthermore, only two independent thermodynamic variables are required. Thus, at equilibrium, Eqs. (A.1) and (A.2) simplify to

$$de = Tds + \frac{P}{\rho^2}\,d\rho \tag{A.7}$$

and

$$de = \left.\frac{\partial e}{\partial s}\right|_{\rho,q=q^*} ds + \left.\frac{\partial e}{\partial \rho}\right|_{s,q=q^*} d\rho \quad . \tag{A.8}$$

Consequently, the equilibrium definitions of the temperature and pressure are the same as the frozen definitions, Eqs. (A.3) and (A.4), except the derivatives are evaluated at $q = q^*$,

$$T = \left.\frac{\partial e}{\partial s}\right|_{\rho,q=q^*} \quad , \tag{A.9}$$

$$P = \rho^2 \left.\frac{\partial e}{\partial \rho}\right|_{s,q=q^*} \quad . \tag{A.10}$$

Cross-differentiation between the two foregoing relations leads to the equilibrium form of Eq. (A.6),

$$\left.\frac{\partial T}{\partial \rho}\right|_{s,q=q^*} = \frac{1}{\rho^2} \left.\frac{\partial P}{\partial s}\right|_{\rho,q=q^*} \quad . \tag{A.11}$$

Proceeding as above, more relations may be obtained from the Gibbs equation expressed in terms of the enthalpy, $h \equiv e + \frac{P}{\rho}$,

$$dh = T ds + \frac{1}{\rho} dP - A\, dq \quad . \tag{A.12}$$

Expressing the enthalpy as a differential quantity in terms of P, s, and q yields

$$dh = \left.\frac{\partial h}{\partial s}\right|_{P,q} ds + \left.\frac{\partial h}{\partial P}\right|_{s,q} dP + \left.\frac{\partial h}{\partial q}\right|_{P,s} dq \quad .$$

Comparison of the above relation with Eq. (A.12) reveals that

$$T = \left.\frac{\partial h}{\partial s}\right|_{P,q} \quad , \tag{A.13}$$

$$\rho = \left.\frac{\partial P}{\partial h}\right|_{s,q} \quad , \tag{A.14}$$

and

$$A = -\left.\frac{\partial h}{\partial q}\right|_{P,s} \quad . \tag{A.15}$$

As with the Gibbs equation expressed in terms of the internal energy, Eqs. (A.13) and (A.14) may be evaluated at equilibrium, that is, at $q = q^*$. This leads to

$$T = \left.\frac{\partial h}{\partial s}\right|_{P,q=q^*} \tag{A.16}$$

and

$$\rho = \left.\frac{\partial P}{\partial h}\right|_{s,q=q^*} \quad . \tag{A.17}$$

A-3 Definitions of Fundamental Thermodynamic Fluid Properties

In this section, some fundamental thermodynamic fluid properties are defined. First to be defined are the frozen forms of the specific heats at constant pressure and constant volume, the isothermal and isentropic bulk moduli, and the coefficient of volume expansion (thermal expansion coefficient). They are, respectively,

$$c_p^\infty \equiv \left.\frac{\partial h}{\partial T}\right|_{P,q} = T\left.\frac{\partial s}{\partial T}\right|_{P,q} \quad , \tag{A.18}$$

$$c_v^\infty \equiv \left.\frac{\partial e}{\partial T}\right|_{\rho,q} = T\left.\frac{\partial s}{\partial T}\right|_{\rho,q} \quad , \tag{A.19}$$

$$\mathcal{K}_T^\infty \equiv \rho\left.\frac{\partial P}{\partial \rho}\right|_{T,q} \quad , \tag{A.20}$$

$$\mathcal{K}_S^\infty \equiv \rho\left.\frac{\partial P}{\partial \rho}\right|_{s,q} \quad , \tag{A.21}$$

$$\alpha_p^\infty \equiv -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_{P,q} \quad , \tag{A.22}$$

The definitions of the equilibrium forms of the above properties, which are denoted by the superscript ^{0}, are identical, except that the partial derivatives are evaluated at $q = q^*$. As noted in the introduction, if the fluid does not contain a relaxation mechanism, the need for the superscript is gone, and it is, therefore, dropped.

The frozen and equilibrium forms of the ratio of specific heats are

$$\gamma^\infty \equiv \frac{c_p^\infty}{c_v^\infty} \tag{A.23}$$

and

$$\gamma^0 \equiv \frac{c_p^0}{c_v^0} \quad . \tag{A.24}$$

It can be shown (see, for example, Thompson (1972, p. 63)) that γ^∞ is equal to the ratio of the squares of the frozen isothermal and isentropic sound speeds, that is, that

$$\left.\frac{\partial P}{\partial \rho}\right|_{s,q} = \gamma^\infty \left.\frac{\partial P}{\partial \rho}\right|_{T,q} \quad . \tag{A.25}$$

A similar relation holds for the equilibrium values. This relation is used in the nondimensionalization of an alternative form of the state equation that involves the temperature. (See the derivation that precedes Eq. (B.23) on page 110.)

A–4 Thermodynamic Identities

In this section, relations between the thermodynamic fluid properties are developed. The first relation developed is

$$T(c^\infty \alpha_p^\infty)^2 \left(\frac{1}{c_v^\infty} - \frac{1}{c_p^\infty}\right) = 1 \quad . \tag{A.26}$$

Equation (A.26) is used to combine the entropy and state equations (see Eq. (2.95) on page 28). Our starting point is the temperature expressed as a perfect differential,

$$dT = \left.\frac{\partial T}{\partial s}\right|_{\rho,q} ds + \left.\frac{\partial T}{\partial \rho}\right|_{s,q} d\rho + \left.\frac{\partial T}{\partial q}\right|_{\rho,s} dq \quad .$$

Dividing through by ds and applying constant P and q yields

$$\left.\frac{\partial T}{\partial s}\right|_{P,q} = \left.\frac{\partial T}{\partial s}\right|_{\rho,q} + \left.\frac{\partial T}{\partial \rho}\right|_{s,q} \left.\frac{\partial \rho}{\partial s}\right|_{P,q} \quad . \tag{A.27}$$

By applying Eq. (A.6) and the defining relation of the frozen sound speed, Eq. (2.60), to the following identity,

$$\left.\frac{\partial \rho}{\partial s}\right|_{P,q} \left.\frac{\partial s}{\partial P}\right|_{\rho,q} \left.\frac{\partial P}{\partial \rho}\right|_{s,q} = -1 \quad ,$$

then rearranging and inserting the results into Eq. (A.27), we obtain the following:

$$\left.\frac{\partial T}{\partial s}\right|_{P,q} = \left.\frac{\partial T}{\partial s}\right|_{\rho,q} - \left(\frac{\rho}{c^\infty}\right)^2 \left(\left.\frac{\partial T}{\partial \rho}\right|_{s,q}\right)^2 \quad .$$

Application of the definitions of the thermal expansion coefficient and the specific heats at constant pressure and volume, Eqs. (A.22), (A.18), and (A.19), respectively, leads to the desired result, Eq. (A.26). Use of an identical procedure leads to the equilibrium counterpart of Eq. (A.26), namely,

$$T(c^0\alpha_p^0)^2\left(\frac{1}{c_v^0}-\frac{1}{c_p^0}\right)=1 \quad . \tag{A.28}$$

For nondimensionalizing the state equation [see the derivation of Eq. (B.20) on page 109], an expression for $\left.\frac{\partial P}{\partial s}\right|_{\rho,q}$ is required. Our starting point is the expansion of frozen specific heat at constant pressure,

$$c_p^\infty \equiv \left.\frac{\partial h}{\partial T}\right|_{P,q} = \left.\frac{\partial h}{\partial s}\right|_{P,q}\left.\frac{\partial s}{\partial \rho}\right|_{P,q}\left.\frac{\partial \rho}{\partial T}\right|_{P,q} \quad .$$

Noting the definitions of the temperature and the thermal expansion coefficient, Eqs. (A.13) and (A.22), respectively, we see that the above relation becomes

$$c_p^\infty = -\rho T\alpha_p^\infty \left.\frac{\partial s}{\partial \rho}\right|_{P,q} \quad . \tag{A.29}$$

Applying the defining relation of the frozen sound speed, Eq. (2.60), to the following identity,

$$\left.\frac{\partial s}{\partial \rho}\right|_{P,q}\left.\frac{\partial \rho}{\partial P}\right|_{s,q}\left.\frac{\partial P}{\partial s}\right|_{\rho,q} = -1 \quad ,$$

and then rearranging and inserting the result into Eq. (A.29) leads to the desired result:

$$\left.\frac{\partial P}{\partial s}\right|_{\rho,q} = \frac{\rho(c^\infty)^2\alpha_p^\infty T}{c_p^\infty} \quad . \tag{A.30}$$

The equilibrium counterpart to Eq. (A.30) is

$$\left.\frac{\partial P}{\partial s}\right|_{\rho,q=q^*} = \frac{\rho(c^0)^2\alpha_p^0 T}{c_p^0} \quad . \tag{A.31}$$

For nondimensionalizing an alternative form of the state equation, an expression for $\left.\frac{\partial P}{\partial T}\right|_{\rho,q}$ is required. (See the derivation that precedes Eq. (B.23) on page 110.) The following identity is our starting point:

$$\left.\frac{\partial P}{\partial T}\right|_{\rho,q}\left.\frac{\partial T}{\partial \rho}\right|_{P,q}\left.\frac{\partial \rho}{\partial P}\right|_{T,q} = -1 \quad .$$

Using the definition of the thermal expansion coefficient, Eq. (A.22), and noting that the ratio of specific heats is equal to the ratio of the isothermal and isentropic bulk moduli, Eq. (A.25), we readily see that

$$\left.\frac{\partial P}{\partial T}\right|_{\rho,q} = \frac{\rho\alpha_p^\infty(c^\infty)^2}{\gamma^\infty} \quad . \tag{A.32}$$

The equilibrium counterpart to Eq. (A.32) is

$$\left.\frac{\partial P}{\partial T}\right|_{\rho,q=q^*} = \frac{\rho\alpha_p^0(c^0)^2}{\gamma^0} \quad . \tag{A.33}$$

A–5 Epilogue

In this appendix, the fundamental relations from thermodynamics that are commonly used in acoustics have been defined (or derived as need be) for fluids with a single relaxation mechanism.

APPENDIX B

ANALYSIS OF THE NONDIMENSIONAL FORMS OF THE BASIC EQUATIONS FOR A HOMOGENEOUS, THERMOVISCOUS FLUID

B–1 Introduction

Developed in this appendix are nondimensional forms of the continuity, momentum, state, and entropy equations that are valid for a homogeneous, thermoviscous fluid that is initially both quiet and irrotational. When nondimensionalizing, it is assumed that the signal is in free space, far from any boundaries. Also developed in this appendix are estimates of the magnitude of the nondimensional coefficients in the equations. Moreover, estimates of the nondimensional signal strength are obtained. The number of assumptions required for small-signal acoustics is then examined. It is noted that only one fundamental assumption is required: That any one of the acoustic field variables be small. It then follows that the others are also small. Next, the nondimensional vorticity equation is derived, and the effect of the irrotional flow assumption is considered. The notation used in this appendix differs somewhat from that in the main text because the analysis is conducted using nondimensional variables. The notation is introduced as needed.

B–2 Nondimensionalization of the Equations of Motion

Developed in this section are the nondimensional forms of the basic equations. First, the dimensional forms of the equations and our notation are introduced. Then, time and distance are nondimensionalized such that their derivatives are equal. Next, the particle velocity and the excess density are nondimensionalized such that the coefficients of the nondimensional continuity equation are unity. The procedure is repeated for the excess pressure while nondimensionalizing the momentum equation. The procedure is repeated again for the

entropy and temperature while nondimensionalizing two different forms of the state equation. Last, the nondimensional entropy equation is obtained.

The equations of motion

The dimensional forms of the continuity, momentum, entropy, and state equations for a homogeneous, thermoviscous fluid are[1]

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \hat{\boldsymbol{\nabla}} \cdot (\hat{\rho} \hat{\mathbf{u}}) = 0 \quad , \tag{B.1}$$

$$\hat{\rho} \frac{D\hat{\mathbf{u}}}{D\hat{t}} = -\hat{\boldsymbol{\nabla}} \hat{P} + \mu \hat{\nabla}^2 \hat{\mathbf{u}} + \left(\mu_{\mathrm{B}} + \tfrac{1}{3}\mu\right) \hat{\boldsymbol{\nabla}}(\hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}}) \quad , \tag{B.2}$$

$$\hat{\rho}\hat{T} \frac{D\hat{s}}{D\hat{t}} = \kappa \hat{\nabla}^2 \hat{T} + \mu_{\mathrm{B}}(\hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}})^2 + \tfrac{1}{2}\mu \left(\frac{\partial \hat{u}_i}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_i} - \tfrac{2}{3}\delta_{ij} \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right)^2 \quad , \tag{B.3}$$

$$\hat{P}(\hat{\rho}, \hat{s}) = P_0 + A\left(\frac{\hat{\rho} - \rho_0}{\rho_0}\right) + (\hat{s} - s_0)\left(\left.\frac{\partial \hat{P}}{\partial \hat{s}}\right|_{\hat{\rho}}\right)_0 + \frac{1}{2!}B\left(\frac{\hat{\rho} - \rho_0}{\rho_0}\right)^2 + \cdots \quad , \tag{B.4}$$

where $\hat{\rho}$ is the density, $\hat{\boldsymbol{\nabla}}$ $(\equiv \mathbf{i}\frac{\partial}{\partial \hat{x}} + \mathbf{j}\frac{\partial}{\partial \hat{y}} + \mathbf{k}\frac{\partial}{\partial \hat{z}})$ is the gradient operator, $\hat{\mathbf{u}}$ is the particle velocity, $\hat{P}$ is the total pressure, $\hat{T}$ is the absolute temperature, $\hat{s}$ is the entropy (per unit mass), μ is the shear viscosity, μ_{B} is the bulk viscosity, κ is the thermal conductivity, $D/D\hat{t}$ is the material derivative,

$$\frac{D(\cdot)}{D\hat{t}} \equiv \frac{\partial(\cdot)}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nabla}}(\cdot) \quad , \tag{B.5}$$

and the constants A and B are given by

$$A \equiv \rho_0 \left(\left.\frac{\partial \hat{P}}{\partial \hat{\rho}}\right|_{\hat{s}}\right)_0 = \rho_0 c_0^2 \tag{B.6}$$

$$B \equiv \rho_0^2 \left(\left.\frac{\partial^2 \hat{P}}{\partial \hat{\rho}^2}\right|_{\hat{s}}\right)_0 \quad . \tag{B.7}$$

The hat symbol ˆ indicates the dimensional form of a quantity. The static value of a quantity, denoted by the subscript 0, does not, however, need a hat ˆ since it is inherently dimensional.

[1] As in the main text, the continuity, momentum, and entropy equations are Eqs. (1.2), (15.6), and (49.5), respectively, in the book by Landau and Lifshitz (1959).

Relationship between the nondimensional time and space derivatives

In this section, time and distance are rendered nondimensional in such a way that the nondimensional time and space derivatives are equal. It is reasonable to do this because it is assumed that the signal is far from any boundaries. Thus, the only dimensions of interest are those associated with the signal.

Time is made dimensionless by dividing by a time that is characteristic of the signal t_{ch} ($\equiv \frac{1}{\hat{\omega}}$ for periodic waves, where $\hat{\omega}$ is the angular frequency),

$$t \equiv \frac{\hat{t}}{t_{ch}} \quad . \tag{B.8}$$

Distance, on the other hand, is nondimensionalized as follows:

$$x_i \equiv \frac{\hat{x}_i}{x_{ch}} \quad ,$$

where x_i is the ith cartesian coordinate and x_{ch} is a characteristic distance. The characteristic distance is to be chosen such that the coefficients in the classical wave equation are unity.[2] Expressing the classical wave equation in terms of nondimensional time and space yields

$$\nabla^2(\cdot) - \left(\frac{x_{ch}}{c_0 t_{ch}}\right)^2 \frac{\partial^2}{\partial t^2} = 0 \quad ,$$

where $\boldsymbol{\nabla}$ is the nondimensional gradient operator. If all the coefficients in the nondimensional wave equation are to be unity, the characteristic distance must be related to the characteristic time by the small-signal sound speed c_0, specifically,

$$x_{ch} \equiv c_0 t_{ch} \quad .$$

Thus, the definition of the nondimensional distance becomes

$$x_i = \frac{\hat{x}_i}{c_0 t_{ch}} \quad . \tag{B.9}$$

Nondimensionalizing the continuity equation

The continuity equation, Eq. (B.1), is now nondimensionalized. As part of the procedure, the particle velocity and excess density, which appear in the

[2]It was pointed out in the main text that the classical wave equation is a good model of acoustic propagation for the important case of small signals in a lossless fluid.

continuity equation, are also rendered nondimensional. The nondimensional particle velocity and excess density are defined as follows:

$$\mathbf{u} \equiv \frac{\hat{\mathbf{u}}}{u_{ch}}$$

and

$$\mathcal{S} \equiv \frac{\hat{\rho} - \rho_0}{\rho_{ch}} \quad ,$$

where u_{ch} and ρ_{ch} are the characteristic particle velocity and density. The nondimensional excess density is referred to as the condensation.[3] The characteristic particle velocity and density are to be chosen in such a way that all the coefficients of the nondimensional continuity equation become unity. Expanding the continuity equation and expressing it in terms of the nondimensional time, distance, excess density, and particle velocity yields

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{u_{ch}}{c_0} \mathcal{S}\, \nabla \cdot \mathbf{u} + \frac{u_{ch}}{c_0} \frac{\rho_0}{\rho_{ch}} (\nabla \cdot \mathbf{u}) + \frac{u_{ch}}{c_0} (\mathbf{u} \cdot \nabla \mathcal{S}) = 0 \quad .$$

If all the coefficients in the above relation are to be unity, then the characteristic particle velocity and density must be defined as follows:

$$u_{ch} \equiv c_0 \quad ,$$

$$\rho_{ch} \equiv \rho_0 \quad .$$

Accordingly, the definitions of the nondimensional particle velocity and excess density become

$$\mathbf{u} = \frac{\hat{\mathbf{u}}}{c_0} \tag{B.10}$$

and

$$\mathcal{S} = \frac{\hat{\rho} - \rho_0}{\rho_0} \quad . \tag{B.11}$$

Thus, the nondimensional continuity equation becomes

$$\frac{\partial \mathcal{S}}{\partial t} + \nabla \cdot [(\mathcal{S} + 1)\mathbf{u}] = 0 \quad . \tag{B.12}$$

[3] The symbol $\mathcal{S}$ has been chosen to denote the condensation rather than the more commonly used s because, in both the main text and this appendix, s is used to denote the specific entropy. Moreover, the condensation is not used outside this appendix.

Nondimensionalizing the momentum equation

The momentum equation, Eq. (B.2), is now rendered dimensionless. Only one new variable, the pressure, is introduced in the momentum equation. The pressure is nondimensionalized such that the coefficients of the nondimensional form of the *lossless* momentum equation become unity. The viscous loss terms in the momentum equation are ignored in this procedure because losses are assumed to be of secondary importance to propagation.

The momentum equation is first rearranged using the following vector identities (Gradshteyn and Ryzhik 1980, Eq. 10.31.7):

$$(\hat{\mathbf{u}}\cdot\hat{\boldsymbol{\nabla}})\hat{\mathbf{u}} = (\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{u}})\times\hat{\mathbf{u}} + \tfrac{1}{2}\hat{\boldsymbol{\nabla}}\hat{u}^2 \quad ,$$

$$\hat{\nabla}^2\hat{\mathbf{u}} = \hat{\boldsymbol{\nabla}}(\hat{\boldsymbol{\nabla}}\cdot\hat{\mathbf{u}}) - \hat{\boldsymbol{\nabla}}\times(\hat{\boldsymbol{\nabla}}\times\hat{\mathbf{u}}) \quad .$$

Inserting the identities into the momentum equation, Eq. (B.2), and using the definition of the vorticity,

$$\hat{\boldsymbol{\Omega}} \equiv \hat{\boldsymbol{\nabla}}\times\hat{\mathbf{u}} \quad , \tag{B.13}$$

leads to

$$\hat{\rho}\left(\frac{\partial\hat{\mathbf{u}}}{\partial\hat{t}} + (\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{u}}) + \tfrac{1}{2}\,\hat{\boldsymbol{\nabla}}\hat{u}^2\right) + \hat{\boldsymbol{\nabla}}\hat{P} = \mu\left[\mathrm{V}\,\hat{\boldsymbol{\nabla}}(\hat{\boldsymbol{\nabla}}\cdot\hat{\mathbf{u}}) - (\hat{\boldsymbol{\nabla}}\times\hat{\boldsymbol{\Omega}})\right] \quad . \tag{B.14}$$

The viscosity number V, which is defined in Eq. (2.82), is indicative of the relative importance of the bulk viscosity to the shear viscosity. In the next section, estimates of the magnitude of the viscosity number and the other nondimensional numbers are obtained.

The excess pressure is nondimensionalized using a characteristic pressure P_{ch} that is to be determined,

$$p' \equiv \frac{\hat{P} - P_0}{P_{ch}} \quad .$$

Introducing the nondimensional time, distance, particle velocity, excess density, and excess pressure into Eq. (B.14) yields

$$(\mathcal{S}+1)\left(\frac{\partial\mathbf{u}}{\partial t} + (\boldsymbol{\Omega}\times\mathbf{u}) + \tfrac{1}{2}\boldsymbol{\nabla}u^2\right) + \frac{P_{ch}}{\rho_0 c_0^2}\,\boldsymbol{\nabla}p' = \mathrm{St}\left[(\boldsymbol{\nabla}\times\boldsymbol{\Omega}) + \mathrm{V}\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right] \quad ,$$

where St is a dimensionless number called the Stokes number (Truesdell 1953, Sec. 3),

$$\mathrm{St} \equiv \frac{\nu}{c_0^2 t_{ch}} \quad . \tag{B.15}$$

(The kinematic vicosity ν is defined in Eq. (2.81).) The magnitude of the Stokes number indicates the importance of viscosity. If the Stokes number is very small (small viscosity or large characteristic time (low frequency)), then the momentum equation may be approximated as

$$(\mathcal{S}+1)\frac{D\mathbf{u}}{Dt}+\frac{P_{ch}}{\rho_0 c_0^2}\nabla p'=0 \quad .$$

If the coefficients of both terms in the foregoing relation are to be unity, then the characteristic pressure must be given by

$$P_{ch}\equiv\rho_0 c_0^2 \quad .$$

Thus, the definition of the nondimensional pressure becomes

$$p'=\frac{\hat{P}-P_0}{\rho_0 c_0^2} \quad . \tag{B.16}$$

Note that $\rho_0 c_0^2$ is the static value of the isentropic bulk modulus.

The nondimensional form of the momentum equation is, accordingly,

$$(\mathcal{S}+1)\left(\frac{\partial\mathbf{u}}{\partial t}+(\boldsymbol{\Omega}\times\mathbf{u})+\tfrac{1}{2}\nabla u^2\right)+\nabla p'=\mathrm{St}\left[\mathrm{V}\,\nabla(\nabla\cdot\mathbf{u})-\nabla\times\boldsymbol{\Omega}\right] \quad . \tag{B.17}$$

Nondimensionalizing the state equation

The state equation, Eq. (B.4), is now nondimensionalized. The only new variable introduced in the state equation is the entropy. The entropy is to be nondimensionalized in such a way that the coefficients of the linearized Taylor series expansion of the state equation are unity. Our starting point is the definition of the nondimensional entropy,

$$s'\equiv\frac{\hat{s}-s_0}{s_{ch}} \quad , \tag{B.18}$$

where s_{ch} is the characteristic value of the entropy that is to be determined. Substituting the above into the linearized state equation noting the defining relation for the sound speed, $c^2\equiv\left.\frac{\partial\hat{P}}{\partial\hat{\rho}}\right|_{\hat{s}}$, yields

$$p'=\mathcal{S}+\frac{s_{ch}}{\rho_0 c_0^2}\left(\left.\frac{\partial\hat{P}}{\partial\hat{s}}\right|_{\hat{\rho}}\right)_0 s' \quad .$$

However, evaluation of Eq. (A.31) from Appendix A at static conditions shows that

$$\left(\left. \frac{\partial \hat{P}}{\partial \hat{s}} \right|_{\hat{\rho}} \right)_0 = \frac{\rho c_0^2 \alpha_{p_0} T_0}{c_{p_0}} \quad ,$$

where α_{p_0} and c_{p_0} are, respectively, the thermal expansion coefficient and the specific heat at constant pressure evaluated at static conditions. Thus, the linearized state equation becomes

$$p' = \mathcal{S} + s_{ch} \frac{\alpha_{p_0} T_0}{c_{p_0}} s' \quad .$$

The coefficient of the linear entropy term is unity if the characteristic entropy is defined as follows:

$$s_{ch} \equiv \frac{c_{p_0}}{\alpha_{p_0} T_0} \quad .$$

Thus, the definition of the nondimensional entropy and the nondimensional equation of state become

$$s' = \frac{\alpha_{p_0} T_0}{c_{p_0}} (\hat{s} - s_0) \tag{B.19}$$

and

$$p' = \mathcal{S} + s' + \frac{1}{2!} \frac{B}{A} \mathcal{S}^2 + \cdots \quad , \tag{B.20}$$

where B/A is a nondimensional number referred to as the parameter of nonlinearity (Beyer 1974, p. 99).

Note that for the special case of an ideal gas, the state equation is $\hat{P} = \hat{\rho} R \hat{T}$, where R is the gas constant. The coefficient of thermal expansion is, accordingly,

$$\alpha_p = \frac{1}{\hat{T}} \quad . \tag{B.21}$$

Thus, for the special case of an ideal gas, it is seen that $\alpha_{p_0} T_0 = 1$ and $s_{ch} = c_{p_0}$.

Nondimensionalizing an alternative state equation

Before nondimensionalizing the entropy equation, an alternative form of the state equation is nondimensionalized,

$$\hat{P} = \hat{P}(\hat{\rho}, \hat{T}) \quad .$$

The reason for nondimensionalizing this equation before the entropy equation is that it motivates the nondimensionalization of the temperature. The nondimensional temperature is defined as follows:

$$T' \equiv \frac{\hat{T} - T_0}{T_{ch}} \quad ,$$

where T_{ch} is a characteristic temperature that is to be determined such that the coefficients in the linearized Taylor series expansion of the alternative state equation are unity. Inserting the nondimensional temperature into the aforementioned equation and then rearranging yields

$$p' = \frac{1}{c_0^2}\left(\left.\frac{\partial \hat{P}}{\partial \hat{\rho}}\right|_{\hat{T}}\right)_0 \mathcal{S} + \frac{T_{ch}}{\rho_0 c_0^2}\left(\left.\frac{\partial \hat{P}}{\partial \hat{T}}\right|_{\hat{\rho}}\right)_0 T' \quad .$$

But, evaluating Eqs. (A.32) and (A.25) from Appendix A at static conditions and inserting them into the above leads to

$$\gamma p' = \mathcal{S} + \alpha_{p_0} T_{ch} T' \quad .$$

The coefficient of T' becomes unity if the characteristic temperature is defined as follows:

$$T_{ch} \equiv \frac{1}{\alpha_{p_0}} \quad .$$

Thus, the definition of the nondimensional temperature and the nondimensional form of the linearized alternate state equation become

$$T' = \alpha_{p_0}(T - T_0) \tag{B.22}$$

and

$$\gamma p' = \mathcal{S} + T' \quad . \tag{B.23}$$

Nondimensionalizing the entropy equation

The entropy equation, Eq. (B.3), is now nondimensionalized. No new variables are introduced in the entropy equation, and the nondimensional form of the entropy equation may, therefore, be obtained directly,

$$(\mathcal{S}+1)(T'+1)\frac{Ds'}{Dt} = \mathrm{St}\,\alpha_{p_0} T_0 \left\{ \frac{1}{\mathrm{Pr}} \nabla^2 T' + \frac{\alpha_{p_0} c_0^2}{c_{p_0}} \left[\frac{\mu_\mathrm{B}}{\mu} (\nabla \cdot \mathbf{u})^2 + \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k} \right)^2 \right] \right\} \quad , \tag{B.24}$$

where Pr is the Prandtl number, which is defined in Eq. (2.98). The Prandtl number is an indicator of the relative importance of viscosity to heat conduction. Note that the coefficient $\alpha_{p_0} c_0^2 / c_{p_0}$ is also a nondimensional number.

B–3 Estimates of the Magnitude of the Nondimensional Coefficients and Signal

Estimates of the magnitude of the nondimensional coefficients in the basic equations are important for the ranking of terms. Estimates of the following nondimensional numbers are, therefore, obtained: (1) the Stokes number, (2) the Prandtl number, (3) the viscosity number, and (4) the nondimensional coefficient $\alpha_{p_0} c_0^2 / c_{p_0}$. Also examined is the magnitude of the nondimensional signal by way of the nondimensional particle velocity.

The nondimensional number that indicates the effects of viscosity is the Stokes number, which is defined in Eq. (B.15). The Stokes number is proportional to the frequency and has the range $0 \leq \mathrm{St} < \infty$. Its value is, therefore, estimated for a relatively high frequency, 1 MHz. Estimates of Stokes number for water and air at 20°C and 1 atm are 2.9×10^{-6} and 8.0×10^{-4}, respectively.[4] It is apparent that, for this temperature, pressure, and frequency, air is more viscous than water. Estimates of the Stokes number at 1 MHz for a variety of other fluids are listed by Truesdell (1953, Table 3.1).

The nondimensional number that indicates the relative importance of viscosity to heat conduction is the Prandtl number, which is defined in Eq. (2.98). From its definition it is clear that the Prandtl number has the range $0 \leq \mathrm{Pr} < \infty$. Estimates of the Prandtl number cited by Truesdell (1953, Table 3.1) indicate it is close to unity for many gases, but generally larger for liquids. In fact, for glycerin, Truesdell cites a Prandtl number of 1000. The AIP Handbook (1977, p. 2–263) cites the Prandtl number for air at 20°C and 1 atm as 0.71. In water for similar conditions, the Prandtl number is about 7.0.[5] Thus, in air at this temperature and pressure, heat conduction is a slightly more important loss mechanism than

[4]The estimate for water is based on density and sound speed data taken from Kinsler et al. (1982, p. 462) and on shear viscosity data from the CRC Handbook (1984, p. F–37): $\rho_0 = 998\ \mathrm{kg/m^3}$, $c_0 = 1481$ m/s, $\mu = 1.002$ centipoises (100 centipoises = 1 poise = 0.1 Pa s). The estimate for air is based on the following data taken from Kinsler et al. (1982, p. 463): $\rho_0 = 1.21\ \mathrm{kg/m^3}$, $c_0 = 343$ m/s, $\mu = 0.0000181$ Pa s.

[5]This value is based on shear viscosity and specific heat at constant pressure data from the CRC Handbook (1984, p. F–37 and D–175) and on coefficient of thermal conduction data from the AIP Handbook (1972, p. 4–151): $\mu = 1.002$ centipoises, $c_{p_0} = 4.2 \times 10^3$ J/kg°C, and $\kappa = 0.6$ W/m°K.

viscosity, whereas in water, heat conduction apparently plays a minor role in attenuation.

Another nondimensional number indicative of loss mechanisms is the viscosity number, which is defined in Eq. (2.82). The viscosity number indicates the relative importance of bulk viscosity to shear viscosity. The range of the viscosity number is $\frac{4}{3} \leq V < \infty$. Truesdell cites the viscosity number for air at standard temperature and pressure as 1.9 and for water at 15 °C and 1 atm as 4.4. Clearly, bulk viscosity plays a more significant role in water than in air.

A nondimensional number that arose in the entropy equation is $\alpha_{p_0} c_0^2 / c_{p_0}$. Note that for the special case of an ideal gas, the following relations hold: $c_{p_0} = R\gamma/(\gamma - 1)$, $\alpha_{p_0} = 1/T_0$, and $c_0^2 = \gamma R T_0$. Thus, we see that

$$\frac{\alpha_{p_0} c_0^2}{c_{p_0}} = \gamma - 1 \quad .$$

Since the value of γ for ideal gases lies in the range $1 < \gamma < 5/3$, the range of $\alpha_{p_0} c_0^2 / c_{p_0}$ for an ideal gas is $0 \leq \alpha_{p_0} c_0^2 / c_{p_0} < 2/3$. For air at 20°C and 1 atm, γ is about 1.4, and the value of $\alpha_{p_0} c_0^2 / c_{p_0}$ in air is therefore about 0.4. The value of $\alpha_{p_0} c_0^2 / c_{p_0}$ in water at 20°C and 1 atm is approximately 0.11.[6]

The acoustic Mach number is a nondimensional number that indicates the magnitude of the signal. The acoustic Mach number is used to help estimate the magnitude of the terms in the basic equations. It is defined as the ratio of the magnitude of the peak particle velocity to the small-signal sound speed,

$$M \equiv \frac{|\hat{\mathbf{u}}_{\mathbf{peak}}|}{c_0} \quad , \tag{B.25}$$

which, in terms of the nondimensional particle velocity, is just $|\mathbf{u}_{\mathbf{peak}}|$. The value of M for sound levels typical of a normal conversion is approximately 1.4×10^{-7}, whereas 3 m from an operating jet engine, M is approximately 1.4×10^{-3}. By definition, the acoustic Mach number is independent of frequency.[7]

[6]This value is based on the previously cited data for the sound speed and specific heat at constant pressure as well as thermal expansion coefficient data from the CRC Handbook (1984, p. F 4): $\alpha_{p_0} = 206 \times 10^{-6}/°K$.

[7]The values cited are based on the values of 60 dB and 140 dB re 20 μPa, respectively, that are cited by Pierce (1981, p. 62).

B–4 The Number of Assumptions Required for Small-Signal Acoustics

From the signal magnitude estimates above, it is clear that for 'normal' acoustics, the acoustic Mach number is a very small number indeed. In fact, it is small even in the case of very loud sounds. The magnitude of the other nondimensional field variables in the basic equations, namely $\mathcal{S}$, p', T', and s', is now examined for the case of small Mach number, $M \ll 1$. It turns out if the acoustic Mach number is small, then the acoustic field variables are of order M or of higher order. Note that this result implies that the number of assumptions required for small-signal acoustics is one, namely, that any one of the field variables be small.

We start by examining the nondimensional continuity equation, Eq. (B.12), with the expectation of obtaining an estimate of the magnitude of the condensation. The three possible solutions are that (1) $|\mathcal{S}| \ll M$, (2) $|\mathcal{S}| \gg M$, and (3) $|\mathcal{S}| \sim M$. The expanded form of Eq. (B.12) is

$$\frac{\partial \mathcal{S}}{\partial t} + \nabla \cdot \mathbf{u} + \mathcal{S}\, \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \mathcal{S} = 0 \quad .$$

Recall that the nondimensional time and space derivatives are equal. An examination of the above equation indicates if the particle velocity is of order M, then second term is of order M, and the third and fourth terms are of order $|\mathcal{S}|M$. The first term is of order $|\mathcal{S}|$. Of the three possible solutions, the only one that balances the equation is the third. In conclusion, it is noted that if $M \ll 1$, then $\mathcal{S} \ll 1$ also. It is further noted that if $M \ll 1$, $\mathcal{S}$ is of order M.

A similar analysis of the momentum equation, Eq. (B.17), is now conducted, and the magnitude of the nondimensional pressure is determined. First note that the definition of the vorticity, Eq. (B.13), indicates that if $M \ll 1$, then the magnitude of the vorticity is also very small, $|\boldsymbol{\Omega}| \ll 1$. (It turns out, as is seen later, that the vorticity is actually quite a bit smaller than M.) Also recall that the Stokes number is generally a small number. Ignoring the terms that are products of small quantities simplifies Eq. (B.17),

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla p' = 0 \quad .$$

Recalling that the nondimensional time and space derivatives are equal, it may be seen that if $M \ll 1$, then $|p'| \ll 1$ also. Thus, p' is of order M, if $M \ll 1$.

It turns out that initially assuming that any one of p', $\mathbf{u}$, or $\mathcal{S}$ is small (much less than unity) implies that the other two are also small. Thus, the

number of assumptions required for small-signal acoustics is one, namely, that one of p', $\mathbf{u}$, or $\mathcal{S}$ be assumed small. That the other two are small follows directly.

Estimates of the magnitude of the nondimensional temperature and entropy fluctuations are now obtained. An examination of the alternate state equation, Eq. (B.23), indicates the nondimensional temperature fluctuation T' is of order $(\gamma - 1)M$. Simplifying the nondimensional entropy equation, Eq. (B.24), by ignoring products of small quantities leads to

$$\frac{\partial s'}{\partial t} = \frac{\mathrm{St}}{\mathrm{Pr}} \alpha_{p_0} T_0 \nabla^2 T' \quad .$$

Thus, s' is of order $\frac{\mathrm{St}}{\mathrm{Pr}} \alpha_{p_0} T_0 (\gamma - 1) M$.

In this section, it was noted that if $M \ll 1$, then the excess pressure and density are of order M. Moreover, it was noted that the number of assumptions required for small-signal acoustics is just one. It was further noted that if $M \ll 1$, then T' and s' are of orders $(\gamma - 1)M$ and $\frac{\mathrm{St}}{\mathrm{Pr}} \alpha_{p_0} T_0 (\gamma - 1) M$, respectively.

B-5 The Nondimensional Vorticity Equation and the Irrotational Flow Assumption

The nondimensional form of the vorticity equation is of interest because it will assist us in understanding the irrotational flow assumption. The nondimensional vorticity equation is formed by taking the curl of the nondimensional momentum equation, Eq. (B.17). This is readily done using index notation such as that described in the book by Panton (1984, Chap. 3).[8] The result is

$$\begin{aligned} \frac{D\boldsymbol{\Omega}}{Dt} &= (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\Omega}(\boldsymbol{\nabla}\cdot\mathbf{u}) \\ &- \frac{1}{(\mathcal{S}+1)^2}(\boldsymbol{\nabla}p' \times \boldsymbol{\nabla}\mathcal{S}) - \mathrm{St}\,\mathrm{V}\left(\frac{1}{\mathcal{S}+1}\right)^2 [\boldsymbol{\nabla}\mathcal{S} \times \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})] \\ &+ \frac{\mathrm{St}}{\mathcal{S}+1}\left[\nabla^2\boldsymbol{\Omega} + \left(\frac{1}{\mathcal{S}+1}\right)[\boldsymbol{\nabla}\mathcal{S} \times (\boldsymbol{\nabla} \times \boldsymbol{\Omega})]\right] \quad . \end{aligned} \tag{B.26}$$

This is an equivalent nondimensional form of Eq. (2.46) in the book by Thompson (1972). The generation and transport of vorticity is now briefly examined. For a more complete discussion, the reader is referred to Thompson and the references contained therein.

[8] Panton (1984, Eq. 13.3.5) uses index notation to develop the dimensional form of the vorticity equation valid for incompressible fluids.

Consider a lossless fluid. Note that the Stokes number is zero for an inviscid fluid, and that, in a uniform lossless fluid, the gradient of the entropy is zero as well. Thus, the term $\nabla p' \times \nabla \mathcal{S}$, which is proportional to $\nabla p' \times \nabla s'$ (recall that $|\nabla p' \times \nabla p'| \equiv 0$), is zero. Accordingly, the vorticity equation simplifies to

$$\frac{D\boldsymbol{\Omega}}{Dt} = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\Omega}(\nabla\cdot\mathbf{u}) \quad .$$

Note if the vorticity is initially zero, the vorticity is always zero because no vorticity is generated in the absence of viscosity or entropy gradients.

In the case of a heat-conducting but inviscid fluid, the Stokes number is still zero, but the gradient of the entropy is nonzero. Equation (B.26) therefore simplifies, but not quite as much as above,

$$\frac{D\boldsymbol{\Omega}}{Dt} = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\Omega}(\nabla\cdot\mathbf{u}) - \frac{1}{(\mathcal{S}+1)^2}(\nabla p' \times \nabla \mathcal{S}) \quad .$$

The new term is a vorticity generation term that is nonzero at time zero in the case of an initially irrotational fluid. Thus, even in the absence of viscosity, vorticity can develop; it does not, however, diffuse. That requires viscosity and is discussed below. If the fluid is initially irrotational, the new term generates vorticity at a rate proportional to $\nabla p' \times \nabla s'$, which, according to our ordering system, is very small—of the order $\frac{\mathrm{St}}{\mathrm{Pr}}\,\alpha_{p_0} T_0(\gamma-1)M^2$. (It may, in fact, be even smaller because a cross-product is proportional to the sine of the angle between the vectors.) The other vorticity generation (or destruction) terms, such as $(\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}$, are of order M higher than the order of $\boldsymbol{\Omega}$.[9] Thus, the vorticity that these terms generate (or destroy) is also very small. However, our nondimensionalization scheme, and hence our ordering system, rest on the assumption that the signal is far from any boundaries. Accordingly, all that may be stated is that the generation of vorticity *in the middle of the fluid* is small. Since the entropy gradient is much sharper at boundaries (due to the presence of the thermal boundary layer, which has not been discussed), vorticity generation at the boundaries is larger.

In a thermoviscous fluid, all terms in the vorticity equation must be retained. Now in the case of an initially irrotational fluid, a second vorticity generation term is nonzero at time zero, $\mathrm{St}\,\mathrm{V}\left(\frac{1}{\mathcal{S}+1}\right)^2[\nabla\mathcal{S}\times\nabla(\nabla\cdot\mathbf{u})]$. The order of this term is $\mathrm{StV}M^2$, again, a very small number *in the middle of the fluid.* Once more it is important to note that near boundaries the gradient terms are much larger, and vorticity generation is also proportionally larger.

[9] Panton (1984, Sec. 13.5) discusses this term at length and points out that it represents the generation or destruction of vorticity by straining vortex lines.

Since the production of vorticity in the middle of the fluid is so small in comparison to that near a boundary, it is of interest to understand the transport of vorticity from the boundaries into the fluid. To that end the leading-order terms in the vorticity equation are examined,

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \mathrm{St}\nabla^2\boldsymbol{\Omega} \quad .$$

Before solving this equation, we place it in dimensional form,

$$\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \hat{t}} = \nu\hat{\nabla}^2\hat{\boldsymbol{\Omega}} \quad .$$

Assuming a solution of the form $\hat{\boldsymbol{\Omega}} = \hat{\mathbf{A}}e^{j(\hat{\omega}\hat{t}-\hat{k}\hat{x})}$ and substituting into the above leads to

$$\hat{k}^2 = -j\frac{\hat{\omega}}{\nu} \quad ,$$

where $\hat{\omega}$ is the angular frequency and $\hat{k}$ is the wave number. The wave number may be defined as follows:

$$\hat{k} \equiv \frac{\hat{\omega}}{c_{\mathrm{ph}}} - j\alpha \quad ,$$

where c_{ph} is the phase velocity and α is the attenuation coefficient that must be positive for stability reasons. Solving for the attenuation coefficient and the phase velocity noting that $\sqrt{j} = \pm(1+j)/\sqrt{2}$ yields

$$\alpha = \frac{1}{c_{\mathrm{ph}}} = \frac{1}{\delta} \quad ,$$

where the negative root was chosen so that α is positive and δ is defined as

$$\delta \equiv \sqrt{2\nu/\hat{\omega}} \quad .$$

Thus, the assumed solution becomes

$$\hat{\boldsymbol{\Omega}} = \hat{\mathbf{A}}\, e^{-\hat{x}/\delta}\, e^{j\hat{\omega}(\hat{t}-\hat{x}/c_{\mathrm{ph}})} \quad .$$

Note that the vorticity decays to $1/e$ of its original value in the distance δ.

Estimates of the distance δ for water and air may be obtained using the viscosity data in footnote 4 on page 111. At a frequency of 1 MHz, δ is about 5.7×10^{-7} m in water and 2.2×10^{-6} m in air. Clearly, even on a wavelength scale, the distance the vorticity penetrates is very small.

In this section, it was noted that the vorticity generation in the middle of the fluid is very small—of order StVM^2 and that the vorticity generated at the boundaries, for which no estimates were provided, is restricted to a very thin layer near the boundary. It is therefore concluded that, away from boundaries, vorticity may be neglected for acoustic problems in which terms of order StVM^2 and higher are neglected.

B–6 Summary

In this appendix, the continuity, momentum, state, and entropy equations that are valid for a homogenous, thermoviscous fluid were nondimensionalized. Also obtained were estimates of the magnitude of the nondimensional coefficients in the equations and of the nondimensional field variables. The number of assumptions required for small-signal acoustics was then analyzed. It was noted that the number is one, namely, that any one of the excess pressure, particle velocity, or excess density be small. Futhermore, the nondimensional vorticity equation was obtained, and it was noted that, away from boundaries, vorticity may be neglected for acoustic problems in which terms of order StVM^2 and higher are neglected.

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